定理1.（列に関する行列式の線形性）

2023.07.19

$$
\begin{array}{c}
 \\
\left|
\begin{array}{ccccc}
a_{11} & \cdots & a_{i1}+b_{i1} & \cdots & a_{1n} \\
a_{12} & \cdots & a_{i2}+b_{i2} & \cdots & a_{2n} \\
\vdots & & \vdots & & \vdots \\
a_{1n} & \cdots & a_{in}+b_{in} & \cdots & a_{nn}
\end{array}
\right|
\end{array}
$$

$$
= \left|
\begin{array}{ccccc}
a_{11} & \cdots & a_{i1} & \cdots & a_{1n} \\
a_{12} & \cdots & a_{i2} & \cdots & a_{2n} \\
\vdots & & \vdots & & \vdots \\
a_{1n} & \cdots & a_{in} & \cdots & a_{nn}
\end{array}
\right|
+
\left|
\begin{array}{ccccc}
a_{11} & \cdots & b_{i1} & \cdots & a_{1n} \\
a_{12} & \cdots & b_{i2} & \cdots & a_{2n} \\
\vdots & & & & \vdots \\
a_{1n} & \cdots & b_{in} & \cdots & a_{nn}
\end{array}
\right|
$$

が成り立つ.

$$
\begin{cases}
a_{i1}+b_{i1} = c_{i1} \\
a_{i2}+b_{i2} = c_{i2} \\
\quad \vdots \\
a_{in}+b_{in} = c_{in}
\end{cases}
$$
とおく.

$$
(\text{左辺}) \underset{\text{テーギ}}{=} \sum_{I=(k_1,\ldots,k_n)} \mathrm{sgn}(I)\, a_{1k_1} a_{2k_2} \cdots c_{ik_i} \cdots a_{nk_n}
$$

$$
= \sum_{I} \mathrm{sgn}(I)\, a_{1k_1} a_{2k_2} \cdots (a_{ik_1} + b_{ik_1}) \cdots a_{nk_n}
$$

$$
= \sum_{I} \mathrm{sgn}(I)\, a_{1k_1} a_{2k_2} \cdots a_{ik_i} \cdots a_{nk_n}
$$

$$
+ \sum_{I} \mathrm{sgn}(I)\, a_{1k_1} a_{2k_2} \cdots b_{ik_i} \cdots a_{nk_n}
$$

$=$ (右辺)

定理2

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & a_{n2} & \cdots & a_{nn} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & \cdots & a_{2n} \\ \vdots & & \vdots \\ a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

n次　　　　$(n-1)$次

(∵)

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$ とする．

$(a_{21} = \cdots = a_{n1} = 0)$

$$(\text{左辺}) = \det A = \sum_{I=(k_1,\cdots,k_n)} \operatorname{sgn}(I)\, a_{1k_1} a_{2k_2} \cdots a_{nk_n}$$

$$= \sum_{I=(1,l_2,\cdots,l_n)} \operatorname{sgn}(I)\, a_{11} a_{2l_2} \cdots a_{nl_n}$$

$$+ \sum_{\substack{J=(k_1,k_2,\cdots,k_n) \\ k_1 \neq 1}} \operatorname{sgn}(J)\, a_{1k_1} a_{2k_2} \cdots a_{nk_n} = 0$$

$$= a_{11} \sum_{I=(1, l_2, \cdots, l_n)} \operatorname{sgn}(I)\, a_{2l_2} \cdots a_{nl_n}$$

$$= a_{11} \sum_{I=(l_2, \cdots, l_n)} \operatorname{sgn}(I)\, a_{2l_2} \cdots a_{nl_n} = (右辺)$$

$(1, 2, \cdots, n-1)$の順列を渉る.

◎ 余因子展開

← n次行列式を$(n-1)$次行列式で表す方法.

(3次で説明)

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11}+0+0 & a_{12} & a_{31} \\ 0+a_{12}+0 & a_{22} & a_{32} \\ 0+0+a_{31} & a_{32} & a_{33} \end{vmatrix}$$

定理1.

$$= \begin{vmatrix} a_{11} & a_{12} & a_{31} \\ 0 & a_{22} & a_{32} \\ 0 & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ 0 & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & a_{31} \\ 0 & a_{22} & a_{32} \\ a_{13} & a_{32} & a_{33} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{31} \\ 0 & a_{22} & a_{32} \\ 0 & a_{32} & a_{33} \end{vmatrix} - \begin{vmatrix} a_{12} & a_{22} & a_{32} \\ 0 & a_{12} & a_{31} \\ 0 & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{13} & a_{32} & a_{33} \\ 0 & a_{12} & a_{31} \\ 0 & a_{22} & a_{32} \end{vmatrix}.$$

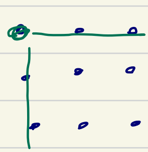
$$= a_{11}\begin{vmatrix} a_{22} & a_{32} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{12} & a_{31} \\ a_{32} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{12} & a_{31} \\ a_{22} & a_{32} \end{vmatrix}$$

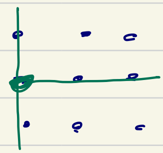
定理2

2次　　2次　　2次.

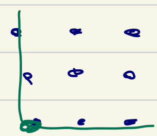
まとめると

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$





$$+ a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$



例

$$\begin{vmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 2 & 1 \end{vmatrix} \xlongequal[①\leftrightarrow②]{} - \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{vmatrix}$$

$$\xlongequal[\substack{②+①\times(-2) \\ ③+①\times(-3)}]{} - \begin{vmatrix} 1 & -1 & 1 \\ 0 & 3 & 1 \\ 0 & 5 & -2 \end{vmatrix} = -1 \begin{vmatrix} 3 & 1 \\ 5 & -2 \end{vmatrix}$$

$$= -\left(3\cdot(-2) - 1\cdot 5\right) = -(-6-5) = 11$$

演

$$\begin{vmatrix} 3 & 2 & 1 \\ -1 & 0 & 2 \\ 1 & 2 & 3 \end{vmatrix}, \quad \begin{vmatrix} -1 & -1 & 2 \\ 3 & 5 & 2 \\ 2 & 4 & 6 \end{vmatrix}$$