2023.07.13

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \ddots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix}$$ $(n \times n)$ 行列 に対して

$$\det A := \sum_{(i_1, i_2, \dots, i_n)} \operatorname{sgn}(i_1, \dots, i_n)\, a_{1 i_1} \cdots a_{n i_n}$$

$(1, 2, \cdots, n)$ の順列 $(i_1, \cdots, i_n)$ のすべてに渡る

順列 $I := (i_1, \dots, i_n)$ に対して

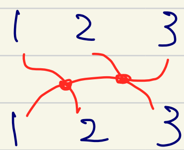
$$\mathrm{inv}(I) := \# \{ (i_a, i_b) \mid 1 \leq a < b \leq n,\ i_a > i_b \}$$

を I の 転倒数 (inversion number) という

てんとうすう

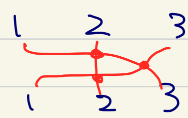
<u>例</u> $n = 3$ のとき.

$$\begin{array}{l} \quad\ \ i_1\ \ i_2\ \ i_3 \\ I_1 = (2, 3, 1) \end{array} \rightsquigarrow \mathrm{inv}(I) = 2$$



$$I_2 = (1\ 2\ 3) \rightsquigarrow \mathrm{inv}(I) = 0$$

$$\begin{array}{l} \quad\ \ i_1\ \ i_2\ \ i_3 \\ I_3 = (3, 2, 1) \end{array} \rightsquigarrow \mathrm{inv}(I) = 3$$



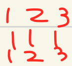
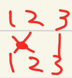
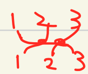
順列 $I := (i_1, \cdots, i_n)$ に対して

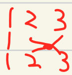
$$\mathrm{sgn}(I) := \begin{cases} +1 & (\mathrm{inv}(I) \text{ が偶数}) \\ -1 & (\mathrm{inv}(I) \text{ が奇数}) \end{cases}$$

<u>例</u>　$\mathrm{sgn}(I_1) = +1$, $\mathrm{sgn}(I_2) = +1$

$\mathrm{sgn}(I_3) = -1$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$ の場合.

$I_1 = (1, 2, 3)$ 
$I_3 = (2, 1, 3)$ 
$I_5 = (3, 1, 2)$ 

$I_2 = (1, 3, 2)$ 
$I_4 = (2, 3, 1)$ 
$I_6 = (3, 2, 1)$ 

$\mathrm{inv}(I_1) = 0$　　$\mathrm{inv}(I_3) = 1$　　$\mathrm{inv}(I_5) = 2$

$\mathrm{inv}(I_2) = 1$　　$\mathrm{inv}(I_4) = 2$　　$\mathrm{inv}(I_6) = 3$

$$\begin{aligned} \det(A) = {} & \mathrm{sgn}(I_1)\, a_{11} a_{22} a_{33} + \mathrm{sgn}(I_2)\, a_{11} a_{23} a_{32} \\ & + \mathrm{sgn}(I_3)\, a_{12} a_{21} a_{33} + \mathrm{sgn}(I_4)\, a_{12} a_{23} a_{31} \\ & + \mathrm{sgn}(I_5)\, a_{13} a_{21} a_{32} + \mathrm{sgn}(I_6)\, a_{13} a_{22} a_{31} \\ = {} & a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} \\ & + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31} \end{aligned}$$

前回の授業ノートと比較せよ

③ 行列式の性質

(1)
$$\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ ca_{i1} & \cdots & ca_{in} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = c \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{i1} & \cdots & a_{in} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$$

1つの行(列)をc倍すると行列式はc倍になる.

(2)
$$\begin{array}{r} \\ \\ i\text{行}\rightarrow \\ \\ j\text{行}\rightarrow \\ \\ \\ \end{array} \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{j1} & \cdots & a_{jn} \\ \vdots & & \vdots \\ a_{i1} & \cdots & a_{in} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = - \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{i1} & \cdots & a_{in} \\ \vdots & & \vdots \\ a_{j1} & \cdots & a_{jn} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$$

2つの行(列)を入れ替えると行列式は −1倍になる.

(3)
$$\begin{array}{r} \\ \\ i\text{行}\rightarrow \\ \\ j\text{行}\rightarrow \\ \\ \\ \end{array}\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{i1}+ca_{j1} & \cdots & a_{in}+ca_{jn} \\ \vdots & & \vdots \\ a_{j1} & \cdots & a_{jn} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{i1} & \cdots & a_{in} \\ \vdots & & \vdots \\ a_{j1} & \cdots & a_{jn} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$$

(列)　　　　(列)

1つの行のc倍を他の行に加えても行列式は変わらない

・ⓘ … i行目
・$\triangle i$ … i列目

例

$$\begin{vmatrix} -1 & -2 & 2 \\ 7 & 1 & 2 \\ 3 & -2 & 5 \end{vmatrix} \underset{\substack{\triangle 2 + \triangle 1 \times (-2) \\ \triangle 3 + \triangle 1 \times 2}}{=\!=} \begin{vmatrix} -1 & 0 & 0 \\ 7 & -13 & 16 \\ 3 & -8 & 11 \end{vmatrix}$$

$$\underset{\substack{②+①\times 7 \\ ③+①\times 3}}{=\!=} \begin{vmatrix} -1 & 0 & 0 \\ 0 & -13 & 16 \\ 0 & -8 & 11 \end{vmatrix} = -1\cdot(-13)\cdot 11 - ((-1)\cdot 16\,(-8))$$

$$= 13\cdot 11 - 16\cdot 8 = 143 - 128 = 15$$