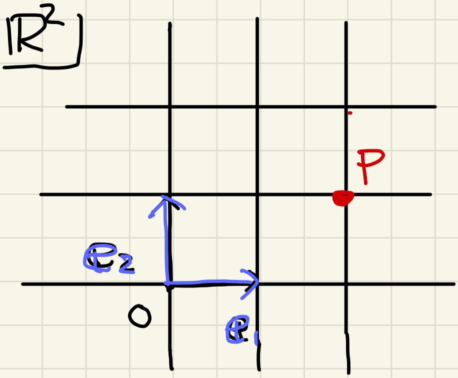
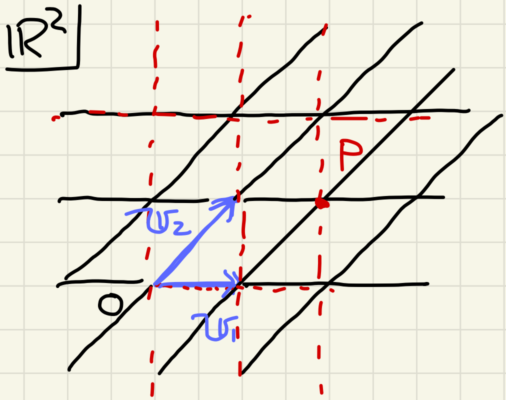
2023/12/19

§. 基底の変換行列.

$$\mathbb{R}^2 = \mathrm{Span}(\mathbb{e}_1, \mathbb{e}_2) = \mathrm{Span}(v_1, v_2) \qquad \left(\mathbb{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \mathbb{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad v_1 = \mathbb{e}_1,\ v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right)$$





$(\mathbb{e}) := (\mathbb{e}_1, \mathbb{e}_2)$ $\qquad$ $(v) := (v_1, v_2)$

$$\overrightarrow{OP} = 2\mathbb{e}_1 + \mathbb{e}_2 \longleftrightarrow \overrightarrow{OP} = x_1 v_1 + x_2 v_2$$

$P : (2, 1)$ $\quad$? $\quad$ $P : (x_1, x_2)$

$$\begin{cases} v_1 = \mathbb{e}_1 \\ v_2 = \mathbb{e}_1 + \mathbb{e}_2 \end{cases} \iff (v_1 \ v_2) = (\mathbb{e}_1 \ \mathbb{e}_2)\underbrace{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}$$

"基底の変換：$(\mathbb{e}) \to (v)$ の行列" という

$$2\mathbf{e}_1+\mathbf{e}_2 = x_1 \mathbf{v}_1 + x_2 \mathbf{v}_2$$

$$\Longleftrightarrow (\mathbf{e}_1\ \mathbf{e}_2)\begin{pmatrix}2\\1\end{pmatrix} = (\mathbf{v}_1\ \mathbf{v}_2)\begin{pmatrix}x_1\\x_2\end{pmatrix}$$

$$= (\mathbf{e}_1\ \mathbf{e}_2)\begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}$$

$$\therefore \begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} \Longleftrightarrow \begin{cases}x_1+x_2=2\\ x_2=1\end{cases} \Longleftrightarrow \begin{cases}x_1=1\\ x_2=1\end{cases}$$

$\rightsquigarrow$ 点 P は $\begin{cases}\{\mathbf{e}_1,\mathbf{e}_2\}\text{に関する座標だと}\ (2,1)\\ \{\mathbf{v}_1,\mathbf{v}_2\}\text{に}\quad\text{〃}\quad (1,1)\end{cases}$

演 $V=\mathbb{R}^3$ とし, V に 2つの基底

$(\mathbf{e}) := (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, $(\mathbf{v}) := \left\{ \mathbf{v}_1 = \begin{pmatrix}-1\\0\\0\end{pmatrix}, \mathbf{v}_2=\begin{pmatrix}1\\1\\0\end{pmatrix}, \mathbf{v}_3=\begin{pmatrix}0\\0\\-2\end{pmatrix}\right\}$

が与えられているとする.

(1) 基底の変換 : $(\mathbf{e}) \to (\mathbf{v})$ の行列を求めよ.

(2) $(\mathbf{e})$ に関する座標で P : $(2,3)$ とする.
点 P の $(\mathbf{v})$ に関する座標を求めよ.

$$\mathbb{R}^2 \ni x_1\mathbf{v}_1 + x_2\mathbf{v}_2 \longmapsto$$

$(\mathbf{v})$

§. 線形写像と基底変換.

- $V = \mathbb{R}^2$, $W = \mathbb{R}^3$.
- $(\mathbb{a}) := \left(\mathbb{a}_1 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \mathbb{a}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$: Vの基底
- $(\mathbb{b}) := \left(\mathbb{b}_1 = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \mathbb{b}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \mathbb{b}_3 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}\right)$: Wの基底
- $f : V \longrightarrow W$: 線形写像

$$\text{∴}A$$

- $$(f(\mathbb{a}_1)\ f(\mathbb{a}_2)) = (\mathbb{b}_1\ \mathbb{b}_2\ \mathbb{b}_3)\begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 0 \end{pmatrix}$$

: fの$(\mathbb{a})$, $(\mathbb{b})$に関する表現行列.

問. VとWの基底を各々

$$(\mathbb{a}') := \left(\mathbb{a}_1' = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \mathbb{a}_2' = \begin{pmatrix} 0 \\ -2 \end{pmatrix}\right)$$

$$(\mathbb{b}') := \left(\mathbb{b}_1' = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \mathbb{b}_2' = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \mathbb{b}_3' = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}\right)$$

に変換するとfの$(\mathbb{a}')$, $(\mathbb{b}')$に関する表現行列はどうなる?

$\rightsquigarrow$ $(f(\mathbb{a}_1'), f(\mathbb{a}_2')) = (\mathbb{b}_1'\ \mathbb{b}_2'\ \mathbb{b}_3')\,A'$

を満たす(3×2)行列Aを求める.

<u>Step 1°</u>. Vの基底変換 : $(\mathbb{a}) \to (\mathbb{a}')$ の行列を求める.

$$\begin{cases} \mathbb{a}_1' = -\mathbb{a}_1 + \mathbb{a}_2 \\ \mathbb{a}_2' = \mathbb{a}_1 - 2\mathbb{a}_2 \end{cases} \Longleftrightarrow (\mathbb{a}_1'\ \mathbb{a}_2') = (\mathbb{a}_1\ \mathbb{a}_2)\underset{\overset{\shortparallel}{B}}{\begin{pmatrix} -1 & 1 \\ 1 & -2 \end{pmatrix}}$$

<u>Step2°</u>. Wの基底変換：$(\mathbb{b}) \to (\mathbb{b}')$の行列を求める.

$$\begin{cases} \mathbb{b}_1' = -\mathbb{b}_3 \\ \mathbb{b}_2' = -\mathbb{b}_2 \\ \mathbb{b}_3' = \mathbb{b}_1 - \mathbb{b}_2 - \mathbb{b}_3 \end{cases} \iff (\mathbb{b}_1' \ \mathbb{b}_2' \ \mathbb{b}_3') = (\mathbb{b}_1 \ \mathbb{b}_2 \ \mathbb{b}_3) \underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & -1 \\ -1 & 0 & -1 \end{pmatrix}}_{= C}$$

<u>Step3°</u>.

$$\begin{aligned}
(f(\mathbb{a}_1') \ f(\mathbb{a}_2')) &= (f(-\mathbb{a}_1 + \mathbb{a}_2), f(\mathbb{a}_1 - 2\mathbb{a}_2)) \\
&= (-f(\mathbb{a}_1) + f(\mathbb{a}_2), f(\mathbb{a}_1) - 2f(\mathbb{a}_2)) \\
&= (f(\mathbb{a}_1) \ f(\mathbb{a}_2)) \underbrace{\begin{pmatrix} -1 & 1 \\ 1 & -2 \end{pmatrix}}_{= B} \\
&= (\mathbb{b}_1 \ \mathbb{b}_2 \ \mathbb{b}_3) A B \\
&= (\mathbb{b}_1' \ \mathbb{b}_2' \ \mathbb{b}_3') C^{-1} A B
\end{aligned}$$

$\therefore$ f の $(\mathbb{a}')$, $(\mathbb{b}')$ に関する表現行列は

$$A' = C^{-1} A B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & -1 \\ -1 & 0 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & -2 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} -1 & 0 & -1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right) \xrightarrow[②+③]{①+③} \left(\begin{array}{ccc|ccc} -1 & 0 & 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right)$$

$$\xrightarrow[②\times(-1)]{①\times(-1)} \left(\begin{array}{ccc|ccc} 1 & & & -1 & 0 & -1 \\ & 1 & & -1 & -1 & 0 \\ & & 1 & 1 & 0 & 0 \end{array}\right) \quad \therefore \ C^{-1} = \begin{pmatrix} -1 & 0 & -1 \\ -1 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$A' = C^{-1} A B = \begin{pmatrix} -1 & 0 & -1 \\ -1 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 0 \\ -1 & -2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ -1 & 3 \\ -1 & 1 \end{pmatrix} //$$

(1) $f(\mathbb{a}_1) = -\mathbb{b}_2$

$f(\mathbb{a}_2) = \mathbb{b}_1 + 2\mathbb{b}_2$

$f(\mathbb{a}_3) = 2\mathbb{b}_1 + 3\mathbb{b}_2$

$$(f(\mathbb{a}_1)\ f(\mathbb{a}_2)\ f(\mathbb{a}_3)) = (\mathbb{b}_1\ \mathbb{b}_2) \begin{pmatrix} 0 & 1 & 2 \\ -1 & 2 & 3 \end{pmatrix}$$

(2) $\mathbb{a}_1' = -2\mathbb{a}_1 - 3\mathbb{a}_2 + \mathbb{a}_3$

$\mathbb{a}_2' = -\mathbb{a}_1 - \mathbb{a}_2$

$\mathbb{a}_3' = \mathbb{a}_1$

$$(\mathbb{a}_1'\ \mathbb{a}_2'\ \mathbb{a}_3') = (\mathbb{a}_1\ \mathbb{a}_2\ \mathbb{a}_3) \begin{pmatrix} -2 & -1 & 1 \\ -3 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

(3) $\mathbf{b}_1' = -\mathbf{b}_1 - \mathbf{b}_2$

$\mathbf{b}_2' = -\mathbf{b}_1$

$$(\mathbf{b}_1' \ \ \mathbf{b}_2') = (\mathbf{b}_1 \ \ \mathbf{b}_2) \begin{pmatrix} -1 & -1 \\ -1 & 0 \end{pmatrix}$$