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Based modules over the ıquantum groups of type AI

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<u>Plan.</u>

1. Background ((ı-)quantum groups, cells)
2. ıquantum group of type AI
3. 1st main thm. (ıcanonical bases)
4. 2nd main thm. (branching rule for $\mathfrak{so}_n \subset \mathfrak{sl}_n$)

1. Background

$\mathfrak{g}$: fin. dim'l semisimple Lie alg. / $\mathbb{C}$

(e.g. $\mathfrak{g} = \mathfrak{sl}_n = \{X \in \mathrm{Mat}_n(\mathbb{C}) \mid \mathrm{tr}\, X = 0\}$)

f.d. $\mathfrak{g}$-mod's are completely reducible

$P^+ := \{$dominant integral weights$\}$

(e.g. $P^+ = \{\lambda = (\lambda_1, \cdots, \lambda_{n-1}) \in \mathbb{Z}_{\geq 0}^{n-1} \mid \lambda_i \geq \lambda_{i+1}\}$)

$\{$f.d. irreducible $\mathfrak{g}$-mod's$\}/_{\text{isom.}} \overset{1:1}{\longleftrightarrow} P^+$

$V(\lambda)$: irr. $\mathfrak{g}$-mod. of highest weight λ $\longleftarrow\!\shortmid$ λ

$\mathfrak{g}$-mod. structure of $V(\lambda)$?

$U_q(\mathfrak{g})$: quantum group / $\mathbb{C}(q)$

= q-deformation of $U(\mathfrak{g})$

($\lim_{q \to 1} U_q(\mathfrak{g}) = U(\mathfrak{g})$: classical limit)

$U(\mathfrak{g})$: universal enveloping alg.

= associative alg. approximation of $\mathfrak{g}$

$\mathfrak{g}$-mod. = $U(\mathfrak{g})$-mod.

$$
\begin{tikzcd}
U_q(\mathfrak{g})\text{-mod} & \\
\mathcal{C} \arrow[u, phantom, sloped, "\subset"] \arrow[r, "\sim"', "\text{classical limit}" below] & \mathfrak{g}\text{-mod}_{f.d.} \\
V_q(\lambda) \arrow[u, phantom, sloped, "\in"] & V(\lambda) \arrow[u, phantom, sloped, "\in"] \arrow[l, mapsto]
\end{tikzcd}
$$

$\uparrow$

irr. $U_q(\mathfrak{g})$-mod. of h.w. λ

$U_q(\mathfrak{g})$-str. of $V_q(\lambda)$?

$V_q(\lambda)$ has a <u>distinguished basis</u>

Lusztig's canonical basis (CB)

= Kashiwara's global crystal basis

$\mathbb{A} := \mathbb{Z}[q, q^{-1}] \subset \mathbb{C}(q)$

$\exists\, U_q(\mathfrak{g})_{\mathbb{A}} \subset U_q(\mathfrak{g})$: free $\mathbb{A}$-mod., $\mathbb{A}$-subalg,

$U_q(\mathfrak{g})_{\mathbb{A}} \otimes_{\mathbb{A}} \mathbb{C}(q) = U_q(\mathfrak{g})$

$V_q(\lambda)_{\mathbb{A}}$:= $\mathbb{A}$-span of the CB.

- $V_q(\lambda)_{\mathbb{A}}$ is a $U_q(\mathfrak{g})_{\mathbb{A}}$-submod.

  free $\mathbb{A}$-mod.

  $V_q(\lambda)_{\mathbb{A}} \otimes_{\mathbb{A}} \mathbb{C}(q) = V_q(\lambda)$

CB $\xrightarrow{q \to \infty}$ crystal basis

↑

combinatorial feature

$\theta : \mathfrak{g} \to \mathfrak{g}$; Lie alg. automorphism s.t. $\theta^2 = \mathrm{id}_{\mathfrak{g}}$

(e.g. $E_{i,j} \mapsto E_{j,i}$ $(i \neq j)$ $E_{i,i} - E_{i+1,i+1} \mapsto -E_{i,i} + E_{i+1,i+1}$)

$\mathfrak{k} := \mathfrak{g}^\theta = \{X \in \mathfrak{g} \mid \theta(X) = X\}$

(e.g. $\mathfrak{k} \simeq \mathfrak{so}_n$)

$(\mathfrak{g}, \mathfrak{k})$ is called a symmetric pair

$U^\iota(\mathfrak{k})$: ιquantum group

$= \begin{cases} \cdot\ q\text{-deformation of } U(\mathfrak{k}) \\ \cdot\ \text{right coideal of } U_q(\mathfrak{g}) \\ \cdot\ \text{max'l among those satisfying} \end{cases}$

$(U_q(\mathfrak{g}), U^\iota(\mathfrak{k}))$ is called quantum symm. pair

Letzter : comprehensive construction of $U^\iota(\mathfrak{k})$

Earlier examples by Noumi and others