2020 Alternative Report of Comprehensive Examination

(From 9am to 5pm, April 16, 2020)

Notes

(1) For all the sheats of your report, write the problem number (i.e. one of $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, $\boxed{4}$) on the upper left, and write your student number and name on the upper right.

(2) Your ability to describe your mathematical demonstration is also evaluated. **Don't write answers on your submitting sheats directly, but do so after making drafts.** (Don't submit the drafts.)

(3) The purpose of the report is to confirm the correctness of your understanding. When you prove something, avoid using the word "obvious" etc., and write the essential point appropriately.

(4) Even if you cannot solve (1) (resp. (2)), you may solve (2) (resp. (3)) by using the conclusion of (1) (resp. (2)).

(5) There are 4 problems, each of which is marked out of 3 points. Thus your total score will be out of 12 points.

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1 Consider the function

$$f(x,y)=\begin{cases}\dfrac{xy^3}{x^2+y^2}\cos\dfrac{1}{x} & (x\neq 0),\\ 0 & (x=0)\end{cases}$$

defined on $\mathbb{R}^2$. Answer the following questions.

(1) Prove that the function $f(x,y)$ is partially differentiable at the origin. Moreover, calculate the partial differential coefficients $f_x(0,0)$ and $f_y(0,0)$.

(2) Prove that for the function $f(x,y)$ there exist constants A, B such that

$$\lim_{(x,y)\to(0,0)}\frac{f(x,y)-f(0,0)-Ax-By}{\sqrt{x^2+y^2}}=0.$$

(3) Calculate $f_x(t,t)$ for $t\neq 0$. Using it, prove that the partial derivative $f_x(x,y)$ with respect to x is not continuous at the origin.

2 Answer the following questions.

(1) Let $\{f_n(x)\}_{n=0}^{\infty}$ be a sequence of real-valued functions on an interval I. Assume $f_n(x)$ are uniformly continuous on I and $\{f_n(x)\}_{n=0}^{\infty}$ converges uniformly to a function $f(x)$ on I. Prove that $f(x)$ is also uniformly continuous on I.

(2) Let a, b, c be real numbers. Give a necessary and sufficient condition for the function $g(x)=ax^2+bx+c$ to be uniformly continuous on $\mathbb{R}$.

(3) Let p, q, r be real numbers. Let $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$, $\{c_n\}_{n=0}^{\infty}$ be sequences of real numbers. Consider the functions

$$h(x)=px^2+qx+r,$$
$$h_n(x)=a_nx^2+b_nx+c_n \ (n\geq 0).$$

Give a necessary and sufficient condition for the sequence $\{h_n(x)\}_{n=0}^{\infty}$ to converge uniformly to $h(x)$ on $\mathbb{R}$.

3 For a $\mathbb{C}$-linear map $f: V\to W$, define the rank of f by $\operatorname{rank} f=\dim_{\mathbb{C}}(\operatorname{Im} f)$. For an $m\times n$ matrix A over $\mathbb{C}$, define the linear map $f_A:\mathbb{C}^n\to\mathbb{C}^m$ by $f_A(x)=Ax$, and the rank of A by $\operatorname{rank} A=\operatorname{rank}(f_A)$. Under these definitions, answer the following questions.

(1) Let A be an $m\times n$ matrix, P an $m\times m$ invertible matrix and Q an $n\times n$ invertible matrix over $\mathbb{C}$. Prove that $\operatorname{rank}(PAQ)=\operatorname{rank} A$.

(2) Assume an $m\times n$ matrix A over $\mathbb{C}$ becomes

$$\begin{pmatrix}E_r & O\\ O & O\end{pmatrix}$$

by elementary row and column operations, where E_r is the $r\times r$ identity matrix and O is a zero matrix. Prove that $\operatorname{rank} A=r$.

(3) Let A be an $m\times n$ matrix over $\mathbb{C}$. Prove that $\operatorname{rank}({}^tA)=\operatorname{rank} A$.

4 Let V be a finite-dimensional vector space over a field K. Let $f: V\to V$ be a linear map. Answer the following questions.

(1) Note that there are ascending and descending chains

$$\operatorname{Ker}(f)\subseteq\operatorname{Ker}(f^2)\subseteq\cdots\subseteq V,$$
$$\cdots\subseteq\operatorname{Im}(f^2)\subseteq\operatorname{Im}(f)\subseteq V$$

of subspaces of V, where f^i denotes the i-th iterate of f. Prove that there exists a positive integer t such that

$$\operatorname{Ker}(f^t)=\operatorname{Ker}(f^{t+i}),$$
$$\operatorname{Im}(f^t)=\operatorname{Im}(f^{t+i})$$

for all positive integers i.

(2) For t as in (1), set $V_0=\operatorname{Ker}(f^t)$ and $V_1=\operatorname{Im}(f^t)$. Prove that $V=V_0\oplus V_1$ holds.

(3) For V_i $(i=0,1)$ as in (2) one has $f(V_i)\subseteq V_i$, and hence one can define the linear map $f|_{V_i}: V_i\to V_i$. Prove that $f|_{V_0}$ is nilpotent and $f|_{V_1}$ is an isomorphism.