Entrance Examination for Master's Program
Graduate School of Mathematics
Nagoya University
2025 Admission

Part 2 of 2

July 28, 2024, 9:00 ∼12:00

Note:

1. Please do not turn pages until told to do so.
2. The problem sheet consists of the cover page and 4 single-sided pages. After the exam has begun, please first confirm that the number of pages and their printing and order are correct. Please report any problem immediately.
3. There are a total of 4 problems labeled **1**, **2**, **3**, and **4**, respectively. Please **answer all 4 problems in English.**
4. The answering sheet consists of 4 pages. Please **confirm the number of pages**, and please **do not remove the staple**.
5. Please write the answers to problems **1**, **2**, **3**, and **4** on pages **1**, **2**, **3**, and **4** of the answering sheet, respectively.
6. **Please write name and seat number in the space provided on each of the** 4 **pages in the answering sheet.**
7. The back side of the 4 pages in the answering sheet may also be used. If used, please check the box at the lower right-hand corner on the front side.
8. If the answering sheet staple is torn, or if additional paper is needed for calculations, please notify the exam proctor.
9. After the exam has ended, please hand in the 4-page answering sheet. The problem sheet and any additional sheets used for calculations may be taken home.

Notation:

The symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of integers, rational numbers, real numbers, and complex numbers, respectively.

1 For N-by-N complex matrices A and B, suppose $A^n = E$ holds and B^n is diagonalizable. Here, n is a positive integer and E is the N-by-N identity matrix.

(1) Let $\omega = \exp\left(\frac{2\pi\sqrt{-1}}{n}\right)$. For a vector $\boldsymbol{v}$ in $\mathbb{C}^N$ and for $i = 0, \ldots, n-1$, define

$$\boldsymbol{v}_i := \frac{1}{n}\sum_{j=0}^{n-1} \omega^{ij} A^j \boldsymbol{v}.$$

Show that $\boldsymbol{v}_i$ is either the zero vector or an eigenvector of A.

(2) Show that for $\boldsymbol{v}$ and $\boldsymbol{v}_0, \ldots, \boldsymbol{v}_{n-1}$ in (1) we have

$$\boldsymbol{v} = \boldsymbol{v}_0 + \cdots + \boldsymbol{v}_{n-1}.$$

Furthermore, show that A is diagonalizable.

(3) Let λ be an eigenvalue of B^n, and let V_λ be the eigenspace of B^n corresponding to λ. Show that if $\boldsymbol{v} \in V_\lambda$, then $B\boldsymbol{v} \in V_\lambda$.

(4) Show that if B is invertible, then B is also diagonalizable.

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(1) Show that for a real number $0 < x < \frac{1}{2}$, we have $x < -\log(1-x) < (2\log 2)x$.

(2) Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers such that $0 < a_n < 1$. Show that

$$\prod_{n=1}^{\infty}(1-a_n) := \lim_{n\to\infty}(1-a_1)(1-a_2)\cdots(1-a_n)$$

converges to a positive real number if and only if $\sum_{n=1}^{\infty} a_n$ converges.

(3) Let $(b_n)_{n=1}^{\infty}$ be a sequence of real numbers such that $0 < b_n < 1$ and $\sum_{n=1}^{\infty}(1-b_n)$ converges. Show that

$$f(x) = \prod_{n=1}^{\infty}\frac{b_n - x}{1 - b_n x}$$

converges locally uniformly on $[0, 1)$. Also, find all points x in $(0, 1)$ satisfying $f(x) = 0$.

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3 Let D be the domain obtained by removing the origin and the negative real axis from the complex plane $\mathbb{C}$. Consider $\log z$ as the single-valued function on D defined by the branch of $\log z$ on D which takes real values on the positive real axis.

(1) Let $0 < \alpha < 2$, and define the complex function f by

$$f(z) = \frac{\log(2+\alpha z)}{z}.$$

Further, let

$$C = \{e^{i\theta} \mid 0 \leq \theta \leq 2\pi\}$$

be a closed curve in the complex plane, and orient the curve C so that the interior of C is on the left as traversing along C. Find $\displaystyle\int_C f(z)\,dz$.

(2) Based on the choice of the branch of $\log z$ in D, show that for points z and w on the closed curve C defined in (1) we have

$$\log\left(4 + 2\alpha(z+w) + \alpha^2 zw\right) = \log(2+\alpha z) + \log(2+\alpha w).$$

(3) Find the value of the integral $\displaystyle\int_0^{2\pi} \log\left(4 + \alpha^2 + 4\alpha\cos\theta\right)\,d\theta$.

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4 For two points a, b in the union of two open intervals $A = (-2, 0) \cup (0, 2)$ contained in the real line $\mathbb{R}$, we write $a \sim b$ if one of the following

(i) $b = a$

(ii) $a \neq \pm 1$ and $b = -a$

holds.

(1) Show that this defines an equivalence relation.

(2) For $a \in A$, the set $[a] = \{b \in A \mid a \sim b\}$ is called the equivalence class of a. Define the set of equivalence classes as $X = A/\sim = \{[a] \mid a \in A\}$ and consider the projection map $p : A \to X;\ a \mapsto [a]$. We introduce a topology on X by defining a subset U of X to be open if $p^{-1}(U)$ is open. Show that the image $p((0, 2))$ of $(0, 2)$ under p is an open set in X and is homeomorphic to $(0, 2)$.

(3) Determine whether X is a Hausdorff space.