Entrance Examination for Master's Program
Graduate School of Mathematics
Nagoya University
2025 Admission

Part 1 of 2

July 27, 2024, 9:00 ∼12:00

Note:

1. Please do not turn pages until told to do so.
2. The problem sheet consists of the cover page and 4 single-sided pages. After the exam has begun, please first confirm that the number of pages and their printing and order are correct. Please report any problem immediately.
3. There are a total of 4 problems labeled [1], [2], [3], and [4], respectively. Please **answer all 4 problems in English.**
4. The answering sheet consists of 4 pages. Please **confirm the number of pages**, and please **do not remove the staple**.
5. Please write the answers to problems [1], [2], [3], and [4] on pages [1], [2], [3], and [4] of the answering sheet, respectively.
6. **Please write name and seat number in the space provided on each of the 4 pages in the answering sheet.**
7. The back side of the 4 pages in the answering sheet may also be used. If used, please check the box at the lower right-hand corner on the front side.
8. If the answering sheet staple is torn, or if additional paper is needed for calculations, please notify the exam proctor.
9. After the exam has ended, please hand in the 4-page answering sheet. The problem sheet and any additional sheets used for calculations may be taken home.

Notation:

The symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of integers, rational numbers, real numbers, and complex numbers, respectively.

1 For the matrix $A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$, let $V = \{\boldsymbol{v} \in \mathbb{R}^4 \mid A\boldsymbol{v} = \boldsymbol{0}\}$. Further, for each real number t, denote by W_t the subspace of $\mathbb{R}^4$ generated by the vectors $\begin{pmatrix} t \\ -t+1 \\ 1 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ t+1 \\ -t+1 \\ -1 \end{pmatrix}$.

(1) Find a necessary and sufficient condition for $V + W_t = \mathbb{R}^4$ in terms of t.

(2) In the case $V + W_t \neq \mathbb{R}^4$, find a basis of $V \cap W_t$. Further, in this case, find a basis of the orthogonal complement of $V + W_t$ in $\mathbb{R}^4$. Here, we define the orthogonal complement with respect to the standard inner product on $\mathbb{R}^4$.

(3) In the case $t = 3$, express the vector $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ as a sum of a vector in V and a vector in W_3.

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2 Let a be a real number, and let A be the matrix given by

$$A = \begin{pmatrix} a & 1 & 1 \\ 0 & a+1 & 1 \\ 0 & 0 & 2a-1 \end{pmatrix}.$$

(1) Find a necessary and sufficient condition for A being diagonalizable in terms of a.

(2) Suppose that the number of distinct eigenvalues of A is 2. Find the Jordan normal form of A, and find an invertible matrix P such that $P^{-1}AP$ is in Jordan normal form.

(3) Suppose that A is not diagonalizable. Find the n-th power A^n of A. Here, n is a positive integer.

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3 (1) For every $a > 0$, determine whether the infinite series

$$\sum_{n=2}^{\infty} \frac{1}{n^a \log n}$$

converges or diverges.

(2) Find all the extremal values of the 2-variable function $f(x, y) = (x^2 + 3y^2)e^{x+y}$ defined on $\mathbb{R}^2$.

(3) Let $D = \{(x, y) \in \mathbb{R}^2 \mid 1 \leq x^2 + y^2 \leq 4,\ x \geq 0\}$. Find the value of the double integral

$$\iint_D x \log(x^2 + y^2)\, dx\, dy.$$

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4 Define the function $f(x)$ on $\mathbb{R}$ as follows:

$$f(x) = \begin{cases} \exp\left(\dfrac{1}{x}\right) & (x < 0) \\ 0 & (x \geq 0) \end{cases}.$$

Let n be a positive integer.

(1) Find the derivative of $f(x)$ for $x < 0$.

(2) Show that $f(x)$ is differentiable at $x = 0$.

(3) Show that for $x < 0$ the n-th derivative $f^{(n)}(x)$ of $f(x)$ can be represented as

$$f^{(n)}(x) = \frac{p_{n-1}(x)}{x^{2n}} f(x)$$

for a polynomial $p_{n-1}(x)$ of degree $(n-1)$.

(4) If the n-th derivative $f^{(n)}(0)$ of the function $f(x)$ at $x = 0$ exists, then compute its value. Further, determine whether the function $f(x)$ can be expanded as a Taylor series at $x = 0$.