Entrance Examination for Master's Program
Graduate School of Mathematics
Nagoya University
2024 Admission

Part 2 of 2

February 4, 2024, 9:00 〜12:00

Note:

1. Please do not turn pages until told to do so.
2. The problem sheet consists of the cover page and 4 single-sided pages. After the exam has begun, please first confirm that the number of pages and their printing and order are correct. Please report any problem immediately.
3. There are a total of 4 problems labeled **1**, **2**, **3**, and **4**, respectively. Please **answer all 4 problems in English.**
4. The answering sheet consists of 4 pages. Please **confirm the number of pages**, and please **do not remove the staple**.
5. Please write the answers to problems **1**, **2**, **3**, and **4** on pages **1**, **2**, **3**, and **4** of the answering sheet, respectively.
6. **Please write name and seat number in the space provided on each of the** 4 **pages in the answering sheet.**
7. The back side of the 4 pages in the answering sheet may also be used. If used, please check the box at the lower right-hand corner on the front side.
8. If the answering sheet staple is torn, or if additional paper is needed for calculations, please notify the exam proctor.
9. After the exam has ended, please hand in the 4-page answering sheet. The problem sheet and any additional sheets used for calculations may be taken home.

Notation:

The symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of integers, rational numbers, real numbers, and complex numbers, respectively.

1 Let V be a real inner product space, where the inner product of $\boldsymbol{v}, \boldsymbol{w} \in V$ is denoted by $\langle \boldsymbol{v}, \boldsymbol{w} \rangle$. For the given vectors $\boldsymbol{v}_1, \cdots, \boldsymbol{v}_n \in V$, define the real numbers $a_{ij} = \langle \boldsymbol{v}_i, \boldsymbol{v}_j \rangle$ and the $n \times n$ matrix $A = (a_{ij})_{1 \le i,j \le n}$.

(1) For $\boldsymbol{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $\boldsymbol{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n$, let

$$\boldsymbol{v}(\boldsymbol{x}) = \sum_{i=1}^{n} x_i \boldsymbol{v}_i, \qquad \boldsymbol{v}(\boldsymbol{y}) = \sum_{i=1}^{n} y_i \boldsymbol{v}_i.$$

Express the inner product $\langle \boldsymbol{v}(\boldsymbol{x}), \boldsymbol{v}(\boldsymbol{y}) \rangle$ in terms of $x_1, \cdots, x_n, y_1, \cdots, y_n$ and a_{ij}.

(2) Show that the eigenvalues of A are all nonnegative real numbers.

(3) Show that A is invertible if and only if $\boldsymbol{v}_1, \cdots, \boldsymbol{v}_n$ are linearly independent.

(4) Show that the dimension of the subspace spanned by $\boldsymbol{v}_1, \cdots, \boldsymbol{v}_n$ equals the rank of A.

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2 Let us express each point in $\mathbb{R}^3$ as $x = (x_1, x_2, x_3)$ and define the norm of x by $\|x\| = \Big(\sum_{k=1}^{3} x_k^2\Big)^{1/2}$. Select and fix one of the positive-valued continuous functions $g(r)$ defined on the interval $[0, \infty)$ in $\mathbb{R}$ such that $g(r)r^2$ is improperly integrable on $[0, \infty)$ and

$$\int_0^\infty g(r) r^2 \, dr = \frac{1}{4\pi}.$$

With this $g(r)$, define the function $G(x, t)$ on $\mathbb{R}^3 \times (0, \infty)$ by

$$G(x, t) = t^{-3} g\Big(\frac{\|x\|}{t}\Big) \qquad (x \in \mathbb{R}^3,\ t > 0).$$

(1) For any $t > 0$, show that $\displaystyle\int_{\mathbb{R}^3} G(x, t) \, dx = 1$.

(2) For any $\delta > 0$, show that $\displaystyle\lim_{t \to 0} \int_{\|x\| \geq \delta} G(x, t) \, dx = 0$.

(3) For a bounded and uniformly continuous function $f(x)$ on $\mathbb{R}^3$, show that

$$u(x, t) = \int_{\mathbb{R}^3} G(y, t) f(x - y) \, dy \qquad (x \in \mathbb{R}^3,\ t > 0)$$

is a function that is bounded and uniformly continuous on $\mathbb{R}^3 \times (0, \infty)$.

(4) Show that, the function $u(x, t)$, defined in (3) with a function $f(x)$ that is bounded and uniformly continuous on $\mathbb{R}^3$, is uniformly convergent to $f(x)$ on $\mathbb{R}^3$ as $t \to 0$.

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3 Consider the complex function f given by $f(z) = \dfrac{e^{iz} - 1}{z}$. For $R > 0$, let

$$C_1 = \{x \mid x \in [0, R]\}, \quad C_2 = \{iy \mid y \in [0, R]\}, \quad C_3 = \left\{ Re^{i\theta} \;\middle|\; \theta \in \left[0, \frac{\pi}{2}\right] \right\}$$

and assume that the closed curve $C = C_1 \cup C_2 \cup C_3$ is oriented so that the interior of C is on the left as traversing along C.

(1) Show that f can be extended to a holomorphic function on $\mathbb{C}$.

(2) Show that $\displaystyle\lim_{R\to\infty} \int_0^{\pi/2} e^{-R\sin\theta}\, d\theta = 0$.

(3) By considering the complex integral of f on C, compute the value of the following improper integral of the real function

$$\int_0^{\infty} \frac{\sin x}{x}\, dx.$$

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4 Let X be a topological space. A subset $A \subset X$ is said to be a connected set in X if there do not exist open sets U, V of X such that

$$U \cap A \neq \emptyset, \quad V \cap A \neq \emptyset, \quad A \subset U \cup V, \quad U \cap V \cap A = \emptyset.$$

Here, $\emptyset$ denotes the empty set.

(1) Let Y be a topological space and consider a continuous map $f : X \to Y$. Show that, if A is a connected set in X, then $f(A)$ is a connected set in Y.

(2) Show that, if A is a connected set in X, then its closure $\overline{A}$ is also a connected set in X.

(3) Let $(A_\lambda)_{\lambda \in \Lambda}$ be a collection of connected sets in X, where Λ is an index set. Show that, if $\bigcap_{\lambda \in \Lambda} A_\lambda$ is nonempty, then $\bigcup_{\lambda \in \Lambda} A_\lambda$ is a connected set in X.