Entrance Examination for Master's Program
Graduate School of Mathematics
Nagoya University
2024 Admission

Part 1 of 2

February 3, 2024, 9:00 ∼12:00

Note:

1. Please do not turn pages until told to do so.
2. The problem sheet consists of the cover page and 4 single-sided pages. After the exam has begun, please first confirm that the number of pages and their printing and order are correct. Please report any problem immediately.
3. There are a total of 4 problems labeled **1**, **2**, **3**, and **4**, respectively. Please **answer all 4 problems in English.**
4. The answering sheet consists of 4 pages. Please **confirm the number of pages**, and please **do not remove the staple**.
5. Please write the answers to problems **1**, **2**, **3**, and **4** on pages **1**, **2**, **3**, and **4** of the answering sheet, respectively.
6. **Please write name and seat number in the space provided on each of the** 4 **pages in the answering sheet.**
7. The back side of the 4 pages in the answering sheet may also be used. If used, please check the box at the lower right-hand corner on the front side.
8. If the answering sheet staple is torn, or if additional paper is needed for calculations, please notify the exam proctor.
9. After the exam has ended, please hand in the 4-page answering sheet. The problem sheet and any additional sheets used for calculations may be taken home.

Notation:

The symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of integers, rational numbers, real numbers, and complex numbers, respectively.

1 Consider the two matrices A and B given by

$$A = \begin{pmatrix} 4 & 6 & -2 \\ 0 & 2 & 0 \\ 3 & 9 & -1 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 6 & -2 \\ 3 & 5 & -2 \\ 12 & 24 & -9 \end{pmatrix}.$$

(1) Show that A and B are both diagonalizable.

(2) Show that A and B are diagonalizable by a common invertible matrix P and obtain such P.

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2 Let t be a real number and consider the vectors

$$\boldsymbol{a} = \begin{pmatrix} -t \\ t \\ -2 \\ 0 \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \quad \boldsymbol{c} = \begin{pmatrix} 1 \\ -t+1 \\ t+1 \\ 1 \end{pmatrix}, \quad \boldsymbol{d} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

in $\mathbb{R}^4$. Let V_1 be the subspace spanned by $\boldsymbol{a}$ and $\boldsymbol{b}$. Also let V_2 be the subspace spanned by $\boldsymbol{c}$ and $\boldsymbol{d}$.

(1) Find the dimensions of V_1 and V_2.

(2) Find the value(s) of t when $V_1 \cap V_2 \neq \{\mathbf{0}\}$.

(3) For the value(s) of t found in (2), give a basis for $V_1 \cap V_2$.

(4) For the value(s) of t found in (2), show that there is an orthonormal basis $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\}$ for $\mathbb{R}^4$ that satisfies the following two conditions

(i) $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ is a basis for V_1 (ii) $\{\boldsymbol{v}_1, \boldsymbol{v}_3\}$ is a basis for V_2.

Furthermore, find such a basis. Here, we assume that $\mathbb{R}^4$ is equipped with the standard inner product.

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3 Answer the following questions.

(1) Compute the value of the integral $\displaystyle\int_0^{0.1} e^{-x^2}\,dx$ by rounding up or down to the fourth decimal place (so that the final answer has four decimal places).

(2) Find the necessary and sufficient condition for $\alpha, \beta > 0$ in order for the improper integral

$$\int_0^{\infty} \frac{e^{\sin x}}{(\sin x + x)^{\alpha}(1 + x^{\beta})}\,dx$$

to converge.

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4 Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences satisfying

$$a_n > 0, \qquad b_n > 0 \qquad (n = 1, 2, \cdots).$$

(1) Suppose that there exists a positive integer N such that for any $n \geq N$

$$\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n}$$

holds. Show that, if the series $\sum_{n=1}^{\infty} b_n$ is convergent, then the series $\sum_{n=1}^{\infty} a_n$ is also convergent.

(2) Let $s > 0$. For $b_n = n^{-s}$, find the limit

$$\lim_{n\to\infty} n\left(\frac{b_{n+1}}{b_n} - 1\right).$$

(3) Suppose that the limit

$$\lim_{n\to\infty} n\left(\frac{a_{n+1}}{a_n} - 1\right) = \alpha$$

exists and is finite. If $\alpha < -1$, then show that the series $\sum_{n=1}^{\infty} a_n$ converges.

Here, you may use, without proof, convergence/divergence of the series $\sum_{n=1}^{\infty} b_n$, where $\{b_n\}_{n=1}^{\infty}$ is the sequence depending on s discussed in (2).