Entrance Examination for Master's Program
Graduate School of Mathematics
Nagoya University
2024 Admission

Part 2 of 2

July 30, 2023, 9:00 ∼12:00

Note:

1. Please do not turn pages until told to do so.
2. The problem sheet consists of the cover page and 4 single-sided pages. After the exam has begun, please first confirm that the number of pages and their printing and order are correct. Please report any problem immediately.
3. There are a total of 4 problems labeled $\boxed{\mathbf{1}}$, $\boxed{\mathbf{2}}$, $\boxed{\mathbf{3}}$, and $\boxed{\mathbf{4}}$, respectively. Please **answer all 4 problems in English.**
4. The answering sheet consists of 4 pages. Please **confirm the number of pages**, and please **do not remove the staple**.
5. Please write the answers to problems $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$ on pages $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$ of the answering sheet, respectively.
6. **Please write name and seat number in the space provided on each of the** 4 **pages in the answering sheet.**
7. The back side of the 4 pages in the answering sheet may also be used. If used, please check the box at the lower right-hand corner on the front side.
8. If the answering sheet staple is torn, or if additional paper is needed for calculations, please notify the exam proctor.
9. After the exam has ended, please hand in the 4-page answering sheet. The problem sheet and any additional sheets used for calculations may be taken home.

Notation:

The symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of integers, rational numbers, real numbers, and complex numbers, respectively.

1 Let $M_n(\mathbb{C})$ be the complex linear space consisting of all $n \times n$ complex matrices. For $A \in M_n(\mathbb{C})$, let

$$Z(A) = \{X \in M_n(\mathbb{C}) \mid AX = XA\}.$$

(1) Suppose that $A, B, P \in M_n(\mathbb{C})$ with P being invertible. Show that, if $B = P^{-1}AP$, then

$$\dim Z(A) = \dim Z(B).$$

(2) Let $A \in M_n(\mathbb{C})$. For an eigenvalue α of A, let $W_\alpha = \ker(A - \alpha I)^n$ be the generalized eigenspace of A with respect to α. Note that I is the $n \times n$ identity matrix. For any $X \in Z(A)$ and any $v \in W_\alpha$, show that $Xv \in W_\alpha$.

(3) Define the $n \times n$ Jordan block J whose diagonal entries are α by

$$J = (J_{ik}), \qquad J_{ik} = \begin{cases} \alpha & (k = i,\ 1 \le i \le n), \\ 1 & (k = i+1,\ 1 \le i \le n-1), \\ 0 & (\text{otherwise}). \end{cases}$$

Let $J = \alpha$ when $n = 1$. For any $\alpha \in \mathbb{C}$, show that $\dim Z(J) = n$.

(4) In a Jordan canonical form of $A \in M_n(\mathbb{C})$, suppose that, for each eigenvalue of A, exactly one Jordan block corresponding to that eigenvalue appears. Show that $\dim Z(A) = n$.

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2 Let f be a real-valued continuous function defined on the interval $[0, \infty)$ and suppose that f is improperly integrable over $[0, \infty)$.

(1) For any $\delta > 0$, show that the sequence $\left\{ \int_{n\delta}^{(n+1)\delta} f(x)\, dx \right\}_{n=0}^{\infty}$ converges to 0 as $n \to \infty$.

(2) Show that there exists a sequence $\{x_n\}_{n=0}^{\infty}$ of positive numbers with $\lim_{n\to\infty} x_n = \infty$ such that

$$\lim_{n\to\infty} f(x_n) = 0.$$

(3) If in addition f is uniformly continuous on $[0, \infty)$, show that

$$\lim_{x\to\infty} f(x) = 0.$$

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3 Consider the complex function f given by

$$f(z) = \frac{\log z}{z^2 + 4}.$$

Here, let D be the domain obtained from the complex plane $\mathbb{C}$ by removing the origin and the imaginary axis in the lower half plane. Then $\log z$ is a single-valued function defined on D if we take the branch of $\log z$ on D so that $\log z$ is real-valued when restricted to the positive real axis. For $R > 2 > \varepsilon > 0$, let

$$C_1 = \{x \mid \varepsilon \leq x \leq R\},$$

$$C_2 = \{Re^{i\theta} \mid 0 \leq \theta \leq \pi\},$$

$$C_3 = \{-y \mid \varepsilon \leq y \leq R\},$$

$$C_4 = \{\varepsilon e^{i\theta} \mid 0 \leq \theta \leq \pi\}$$

and assume that the closed curve $C = C_1 \cup C_2 \cup C_3 \cup C_4$ in D is oriented so that the interior of C is on the left as traversing along C.

(1) For $z = re^{i\theta}$ $(r > 0, -\frac{\pi}{2} < \theta < \frac{3\pi}{2})$, express $\log z$ in terms of r and θ.

(2) Find the value of the complex integral $\displaystyle\int_C f(z)dz$.

(3) Show that $\displaystyle\lim_{R\to\infty} \int_{C_2} f(z)dz = 0$ and $\displaystyle\lim_{\varepsilon\to 0} \int_{C_4} f(z)dz = 0$.

(4) Find the value of the real integral $\displaystyle\int_0^\infty \frac{\log x}{x^2 + 4} dx$.

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4 Answer the following questions.

(1) Show that a closed set A in a compact topological space X is a compact set.

(2) Show that a compact set B in a Hausdorff topological space Y is a closed set.

(3) Let X, Y be topological spaces. Show that, if f is a continuous map from X to Y and A is a compact set in X, then the image $f(A)$ is a compact set in Y.

(4) Show that, if f is a continuous bijection from a compact topological space X to a Hausdorff topological space Y, then f is a homeomorphism.