Entrance Examination for Master's Program
Graduate School of Mathematics
Nagoya University
2024 Admission

Part 1 of 2

July 29, 2023, 9:00 ∼12:00

Note:

1. Please do not turn pages until told to do so.
2. The problem sheet consists of the cover page and 4 single-sided pages. After the exam has begun, please first confirm that the number of pages and their printing and order are correct. Please report any problem immediately.
3. There are a total of 4 problems labeled $\boxed{\mathbf{1}}$, $\boxed{\mathbf{2}}$, $\boxed{\mathbf{3}}$, and $\boxed{\mathbf{4}}$, respectively. Please **answer all 4 problems in English.**
4. The answering sheet consists of 4 pages. Please **confirm the number of pages**, and please **do not remove the staple**.
5. Please write the answers to problems $\boxed{\mathbf{1}}$, $\boxed{\mathbf{2}}$, $\boxed{\mathbf{3}}$, and $\boxed{\mathbf{4}}$ on pages $\boxed{\mathbf{1}}$, $\boxed{\mathbf{2}}$, $\boxed{\mathbf{3}}$, and $\boxed{\mathbf{4}}$ of the answering sheet, respectively.
6. **Please write name and seat number in the space provided on each of the** 4 **pages in the answering sheet.**
7. The back side of the 4 pages in the answering sheet may also be used. If used, please check the box at the lower right-hand corner on the front side.
8. If the answering sheet staple is torn, or if additional paper is needed for calculations, please notify the exam proctor.
9. After the exam has ended, please hand in the 4-page answering sheet. The problem sheet and any additional sheets used for calculations may be taken home.

Notation:

The symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of integers, rational numbers, real numbers, and complex numbers, respectively.

1 Let V and W_t be the subspaces of $\mathbb{R}^4$ spanned by the two vectors

$$\begin{pmatrix} 1 \\ 0 \\ -1 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ -2 \\ -1 \end{pmatrix}$$

and the three vectors

$$\begin{pmatrix} t \\ 1 \\ t+2 \\ t-1 \end{pmatrix}, \quad \begin{pmatrix} t+1 \\ -1 \\ t+1 \\ t+1 \end{pmatrix}, \quad \begin{pmatrix} t-1 \\ 0 \\ t \\ t-2 \end{pmatrix},$$

respectively, where t is a real parameter.

(1) Assuming the standard inner product in $\mathbb{R}^4$, find the dimension of the orthogonal complement $V^{\perp}$ of V in $\mathbb{R}^4$. Also, find a basis for $V^{\perp}$.

(2) Find a system of linear equations whose solution space equals V.

(3) Find a system of linear equations whose solution space equals W_t.

(4) Find the dimension and a basis for the subspace $V \cap W_t$.

 (continued on next page)

2 Let $V = \{a + bx + cx^2 + dx^3 \mid a, b, c, d \in \mathbb{R}\}$ be the real linear space consisting of all polynomials in x of degree at most 3. For a real number α, define the linear map $\Phi_\alpha : V \to \mathbb{R}^4$ by

$$\Phi_\alpha(f(x)) = \begin{pmatrix} f(\alpha) \\ f'(\alpha) \\ f''(\alpha) \\ f'''(\alpha) \end{pmatrix}, \qquad f(x) \in V.$$

Note here that the second, third, and fourth components are the first, second, and third derivatives of f at $x = \alpha$, respectively.

(1) Find the representation matrix A_α of Φ_α with respect to the basis $\{1, x, x^2, x^3\}$ for V and the standard basis for $\mathbb{R}^4$.

(2) Verify that the matrix A_α obtained in (1) is invertible. Furthermore, find its inverse A_α^{-1}.

(3) Let $\alpha \neq \beta$. Find the Jordan canonical form of the matrix $A_\alpha^{-1} A_\beta$. (It is not necessary to find an invertible matrix that actually transforms $A_\alpha^{-1} A_\beta$ into the Jordan canonical form.)

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3 Answer the following questions.

(1) Taking the Euclidean distance in $\mathbb{R}^2$, find the distance between the curve $x^3 + 2y^3 = 10$ and the point $(0, 0)$.

(2) Find the volume of the solid in $\mathbb{R}^3$ given by

$$\{(x, y, z) \in \mathbb{R}^3 \mid e^z \leq x^2 + y^2 \leq (e-1)z + 1\}.$$

(3) For the rational function on $\mathbb{R}$ given by

$$f(x) = \frac{1}{2x^2 - 3x + 1},$$

find the Taylor series expansion around $x = 0$ and its radius of convergence.

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4 Let $\rho \geq 1$ and consider the domain

$$D_\rho = \{(x, y) \in \mathbb{R}^2 \mid f(x, y) > \rho^2\}$$

in $\mathbb{R}^2$, where $f(x, y)$ is the quadratic form given by

$$f(x, y) = 5x^2 + 8xy + 5y^2.$$

Furthermore, define the improper integral J_ρ over D_ρ with parameters α, $\beta > 0$ by

$$J_\rho = \iint_{D_\rho} \big(f(x, y) - 1\big)^\alpha f(x, y)^{-\alpha-1} \big(\log f(x, y)\big)^{-\beta} \, dx \, dy.$$

(1) Find a 2×2 real symmetric matrix A such that

$$f(x, y) = \begin{pmatrix} x & y \end{pmatrix} A \begin{pmatrix} x \\ y \end{pmatrix}$$

for any $(x, y) \in \mathbb{R}^2$. In addition, find the eigenvalues of A.

(2) Show that whether J_3 is convergent or divergent does not depend on α. Furthermore, find necessary and sufficient conditions on β so that J_3 converges.

(3) Find necessary and sufficient conditions on α, β so that J_1 converges.