Entrance Examination for Master's Program
Graduate School of Mathematics
Nagoya University
2023 Admission

Part 2 of 2

July 31, 2022, 9:00 ∼12:00

Note:

1. Please do not turn pages until told to do so.
2. The problem sheet consists of the cover page and 4 single-sided pages. After the exam has begun, please first confirm that the number of pages and their printing and order are correct. Please report any problem immediately.
3. There are a total of 4 problems labeled **1**, **2**, **3**, and **4**, respectively. Please **answer all 4 problems.**
4. The answering sheet consists of 4 pages. Please **confirm the number of pages**, and please **do not remove the staple**.
5. Please write the answers to problems **1**, **2**, **3**, and **4** on pages **1**, **2**, **3**, and **4** of the answering sheet, respectively.
6. **Please write name and seat number in the space provided on each of the** 4 **pages in the answering sheet.**
7. The back side of the 4 pages in the answering sheet may also be used. If used, please check the box at the lower right-hand corner on the front side.
8. If the answering sheet staple is torn, or if additional paper is needed for calculations, please notify the exam proctor.
9. After the exam has ended, please hand in the 4-page answering sheet. The problem sheet and any additional sheets used for calculations may be taken home.

Notation:

The symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of integers, rational numbers, real numbers, and complex numbers, respectively.

1 Let V be a finite dimensional real vector space and let $\langle \boldsymbol{v}, \boldsymbol{w} \rangle$ $(\boldsymbol{v}, \boldsymbol{w} \in V)$ denote the inner product in V. A linear map $p : V \to V$ is said to be a projection if $p^2 = p$. Moreover, such p is said to be an orthogonal projection if $\operatorname{Ker} p$ and $\operatorname{Im} p$ are orthogonal.

(1) Suppose that $p : V \to V$ is an orthogonal projection. Show that

$$\langle p(\boldsymbol{v}), \boldsymbol{v} - p(\boldsymbol{v}) \rangle = 0$$

for every $\boldsymbol{v} \in V$.

(2) Suppose that $p : V \to V$ is an orthogonal projection. Show that $|p(\boldsymbol{v})| \leq |\boldsymbol{v}|$ for every $\boldsymbol{v} \in V$. Here, $|\boldsymbol{x}| := \sqrt{\langle \boldsymbol{x}, \boldsymbol{x} \rangle}$.

(3) Assume that p is a projection and $|p(\boldsymbol{v})| \leq |\boldsymbol{v}|$ for every $\boldsymbol{v} \in V$. Show that

$$|\boldsymbol{u}|^2 + 2t\langle \boldsymbol{u}, \boldsymbol{w} \rangle \geq 0$$

for every $\boldsymbol{u} \in \operatorname{Ker} p$, $\boldsymbol{w} \in \operatorname{Im} p$ and every $t \in \mathbb{R}$.

(4) Under the assumptions in (3), show that p is an orthogonal projection.

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2 Let f and g be continuous real-valued functions on the interval $[0, \infty)$ and suppose that both f^2 and g^2 are improperly integrable on $[0, \infty)$.

(1) Let $m > 0$. By integrating the function $(|f(x)||g(y)| - |f(y)||g(x)|)^2$ of two variables x and y on $[0, m] \times [0, m]$, show that

$$\left(\int_0^m |f(x)g(x)|\, dx \right)^2 \leq \int_0^m f(x)^2\, dx \int_0^m g(x)^2\, dx$$

holds.

(2) Show that $u(x) = \displaystyle\int_0^\infty f(x+y)g(y)\, dy$ is a bounded function on $[0, \infty)$.

(3) For an arbitrary positive integer m, consider the function

$$u_m(x) = \int_0^m f(x+y)g(y)\, dy$$

on $[0, \infty)$. For any real positive number L, show that $u_m(x)$ is uniformly continuous on $[0, L]$.

(4) Show that the sequence $\{u_m\}$ of functions in (3) converges uniformly on $[0, \infty)$ to the function u in (2).

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3 Answer the following questions.

(1) For a real number θ, let $z = e^{i\theta}$. Express the function

$$f(\theta) = \frac{1}{33 - 40\cos\theta + 8\cos 2\theta}$$

as a function of z.

(2) Compute the value of the integral $\displaystyle\int_0^{2\pi} f(\theta)\,d\theta$.

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4 Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous map.

(1) Suppose that f satisfies the condition

(∗) For every bounded subset K of $\mathbb{R}$, its preimage $f^{-1}(K)$ is a bounded set.

Show that f is a closed map, that is, for every closed set F its image $f(F)$ is a closed set.

(2) Show that f satisfies the condition (∗) above if and only if $\lim_{x \to \pm\infty} |f(x)| = \infty$.