Entrance Examination for Master's Program
Graduate School of Mathematics
Nagoya University
2022 Admission

Part 2 of 2

February 6, 2022, 9:00 ∼12:00

Note:

1. Please do not turn pages until told to do so.
2. The problem sheet consists of the cover page and 4 single-sided pages. After the exam has begun, please first confirm that the number of pages and their printing and order are correct. Please report any problem immediately.
3. There are a total of 4 problems labeled [1], [2], [3], and [4], respectively. Please **answer all 4 problems.**
4. The answering sheet consists of 4 pages. Please **confirm the number of pages**, and please **do not remove the staple**.
5. Please write the answers to problems [1], [2], [3], and [4] on pages [1], [2], [3], and [4] of the answering sheet, respectively.
6. **Please write name and seat number in the space provided on each of the 4 pages in the answering sheet.**
7. The back side of the 4 pages in the answering sheet may also be used. If used, please check the box at the lower right-hand corner on the front side.
8. If the answering sheet staple is torn, or if additional paper is needed for calculations, please notify the exam proctor.
9. After the exam has ended, please hand in the 4-page answering sheet. The problem sheet and any additional sheets used for calculations may be taken home.

Notation:

The symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of integers, rational numbers, real numbers, and complex numbers, respectively.

1 Let V be a finite-dimensional real vector space. Let V^* be its dual space, i.e., V^* is the real vector space defined as the set consisting of all linear maps from V to $\mathbb{R}$, with the operations $(\phi+\psi)(v) = \phi(v)+\psi(v)$, $(c\phi)(v) = c\phi(v) \quad (\phi, \psi \in V^*, v \in V, c \in \mathbb{R})$. Next, for a basis $\{e_k \,|\, k \in I\}$ of V, define $e_j^* \in V^*$ for each $j \in I$ by the following condition.

$$e_j^*(e_k) = \begin{cases} 1 & (j = k) \\ 0 & (j \neq k) \end{cases}$$

(1) Show that $\{e_j^* \,|\, j \in I\}$ is a basis of V^*.

(2) For any element v of V, show that $v = \sum_{k \in I} e_k^*(v) e_k$.

Below, V denotes the real vector space consisting of all polynomials in the variable x of degree at most n with real coefficients, and $I = \{0, 1, \ldots, n\}$, $e_k = x^k$ $(k \in I)$.

(3) For $v = p(x) \in V$, show that $e_k^*(v) = \dfrac{1}{k!} p^{(k)}(0)$, where $p^{(k)}(x)$ represents the k-th derivative of the polynomial $p(x)$.

(4) For $v = p(x) \in V$, define $\hat{v} \in V^*$ as

$$\hat{v}(u) = \int_0^1 p(x) q(x) \, dx \qquad (u = q(x) \in V).$$

Find $a_j \in \mathbb{R}$ $(j \in I)$ such that $\hat{v} = \sum_{j \in I} a_j e_j^*$ and express them using $p^{(k)}(0)$ $(k \in I)$.

 (continued on next page)

2 Consider the sequence $\{\phi_n\}_{n=1}^{\infty}$ of functions over $[0, \infty)$ defined as $\phi_n(x) = \dfrac{n}{1 + n^2 x^2}$.

(1) Find the value of $\displaystyle\int_0^{\infty} \phi_n(x)\, dx$.

(2) For any $\delta > 0$, show that

$$\lim_{n \to \infty} \int_{\delta}^{\infty} \phi_n(x)\, dx = 0$$

holds.

(3) Let $f : [0, \infty) \to \mathbb{R}$ be bounded and continuous. Then show that

$$\lim_{n \to \infty} \int_0^{\infty} f(x)\phi_n(x)\, dx = \frac{\pi}{2} f(0)$$

holds.

(continued on next page)

3 Consider the complex function

$$f(z) = \frac{e^{iz^2}}{z}.$$

For $R > \varepsilon > 0$, define

$$C_1 = \{r \mid \varepsilon \le r \le R\} \qquad C_2 = \{Re^{i\theta} \mid 0 \le \theta \le \frac{\pi}{4}\}$$

$$C_3 = \{re^{\frac{\pi}{4}i} \mid \varepsilon \le r \le R\} \qquad C_4 = \{\varepsilon e^{i\theta} \mid 0 \le \theta \le \frac{\pi}{4}\}$$

and set their orientations so that the closed curve $C = C_1 \cup C_2 \cup C_3 \cup C_4$ is oriented counterclockwise.

(1) Find the minimum of $\dfrac{\sin(2\theta)}{\theta}$ for $0 < \theta \le \dfrac{\pi}{4}$.

(2) Show that $\displaystyle\lim_{R\to\infty} \int_{C_2} f(z)\, dz = 0$.

(3) Show that $\displaystyle\lim_{\varepsilon\to 0} \int_{C_4} f(z)\, dz = -\frac{\pi}{4}i$.

(4) Show that $\displaystyle\int_{C_3} f(z)\, dz$ has real value.

(5) Find the value of the improper integral

$$\int_0^\infty \frac{\sin x^2}{x}\, dx$$

and explain how you obtained it.

(continued on next page)

4 Let X be a topological space. A function $f : X \to \mathbb{R}$ over X is said upper semicontinuous if for any $\lambda \in \mathbb{R}$, the set $U_\lambda = f^{-1}\big((-\infty, \lambda)\big)$ is an open set of X. Answer the following questions.

(1) Find if the function

$$g(x) = \begin{cases} \sqrt{x} + 1 & (x \geq 0) \\ -x & (x < 0) \end{cases}$$

over $\mathbb{R}$ is upper semicontinuous and explain the reason.

(2) Show that when X is compact, an upper semicontinuous function $f : X \to \mathbb{R}$ is bounded above.

(3) Assume that X is compact and f is upper semicontinuous, and write $\alpha = \sup_{x \in X} f(x)$. Show that there exists $x_0 \in X$ such that $\alpha = f(x_0)$.