

**Entrance Examination for Master's Program  
Graduate School of Mathematics  
Nagoya University  
2022 Admission**

**Part 1 of 2**

February 5, 2022, 9:00 ~12:00

**Note:**

1. Please do not turn pages until told to do so.
2. The problem sheet consists of the cover page and 4 single-sided pages. After the exam has begun, please first confirm that the number of pages and their printing and order are correct. Please report any problem immediately.
3. There are a total of 4 problems labeled **1**, **2**, **3**, and **4**, respectively. Please **answer all 4 problems**.
4. The answering sheet consists of 4 pages. Please **confirm the number of pages**, and please **do not remove the staple**.
5. Please write the answers to problems **1**, **2**, **3**, and **4** on pages **1**, **2**, **3**, and **4** of the answering sheet, respectively.
6. **Please write name and seat number in the space provided on each of the 4 pages in the answering sheet.**
7. The back side of the 4 pages in the answering sheet may also be used. If used, please check the box at the lower right-hand corner on the front side.
8. If the answering sheet staple is torn, or if additional paper is needed for calculations, please notify the exam proctor.
9. After the exam has ended, please hand in the 4-page answering sheet. The problem sheet and any additional sheets used for calculations may be taken home.

**Notation:**

The symbols  $\mathbb{Z}$ ,  $\mathbb{Q}$ ,  $\mathbb{R}$ , and  $\mathbb{C}$  denote the sets of integers, rational numbers, real numbers, and complex numbers, respectively.

**1** For a real number  $c$ , let  $V$  be the subspace of  $\mathbb{R}^4$  generated by the three vectors

$$\begin{pmatrix} c+2 \\ c-2 \\ c \\ c-2 \end{pmatrix}, \quad \begin{pmatrix} c+1 \\ c-2 \\ c-1 \\ c-1 \end{pmatrix}, \quad \begin{pmatrix} c \\ c \\ c-2 \\ c \end{pmatrix},$$

and  $W$  be the subspace generated by the two vectors

$$\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}.$$

- (1) Find the dimension of  $V$ .
- (2) Find the dimension of  $V + W$ .
- (3) Find the dimension of  $V \cap W$ .

**2** Consider the following matrix  $A$ , where  $t$  is a real number.

$$A = \begin{pmatrix} t^2 + 1 & 0 & -1 \\ 0 & -t^2 - \frac{1}{2} & 0 \\ 1 & 0 & -t^2 - 1 \end{pmatrix}$$

- (1) Find the rank of  $A$ .
- (2) Find the eigenvalues of  $A$ .
- (3) Find necessary and sufficient conditions on  $t$  so that  $A$  is diagonalizable.

**3** Answer the following questions about the function

$$f(x, y) = \sin x + \sin y - \sin(x + y) \quad (-\pi \leq x < \pi, -\pi \leq y < \pi).$$

(1) Find a polynomial  $P(x, y)$  in  $x$  and  $y$  of degree 3 such that

$$\lim_{(x,y) \rightarrow (0,0)} \frac{f(x, y) - P(x, y)}{(x^2 + y^2)^{\frac{3}{2}}} = 0.$$

(2) Find the extrema of  $f(x, y)$  and the values  $x, y$  achieving the extrema.

**4** Answer the following questions.

(1) Let  $a$  be a real number. Consider the domain  $D$  inside  $\mathbb{R}^3$  defined as

$$D = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \leq 1, 1 - x^2 - y^2 \leq z\}.$$

Give necessary and sufficient conditions on  $a$  so that the improper integral

$$\iiint_D \frac{1}{z^a} dx dy dz$$

converges, and give the value of the integral in this case.

(2) Compute the value of

$$\iint_D |x - y| dx dy,$$

where  $D = \{(x, y) \in \mathbb{R}^2 \mid \sqrt{x} + \sqrt{y} \leq 1, x \geq 0, y \geq 0\}$ .