Entrance Examination for Master's Program
Graduate School of Mathematics
Nagoya University
2022 Admission

Part 2 of 2

August 1, 2021, 9:00 ～12:00

Note:

1. Please do not turn pages until told to do so.

2. The problem sheet consists of the cover page and 4 single-sided pages. After the exam has begun, please first confirm that the number of pages and their printing and order are correct. Please report any problem immediately.

3. There are a total of 4 problems labeled **1**, **2**, **3**, and **4**, respectively. Please **answer all 4 problems.**

4. The answering sheet consists of 4 pages. Please **confirm the number of pages**, and please **do not remove the staple**.

5. Please write the answers to problems **1**, **2**, **3**, and **4** on pages **1**, **2**, **3**, and **4** of the answering sheet, respectively.

6. **Please write name and application number in the space provided on each of the 4 pages in the answering sheet.**

7. The back side of the 4 pages in the answering sheet may also be used. If used, please check the box at the lower right-hand corner on the front side.

8. If the answering sheet staple is torn, or if additional paper is needed for calculations, please notify the exam proctor.

9. After the exam has ended, please hand in the 4-page answering sheet. The problem sheet and any additional sheets used for calculations may be taken home.

Notation:

The symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of integers, rational numbers, real numbers, and complex numbers, respectively.

1 Let A be an $n \times n$ complex matrix satisfying the condition $P^2 = P$, where $P = A^*A$ and $A^* = {}^t\bar{A}$ (the transpose of the matrix in which each entry is the complex conjugate of the corresponding entry of A). Define the standard Hermitian inner product of two elements $\boldsymbol{x} = {}^t(x_1, \ldots, x_n)$, $\boldsymbol{y} = {}^t(y_1, \ldots, y_n)$ in $\mathbb{C}^n$ by $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \sum_{i=1}^n x_i \bar{y}_i$ and let $|\boldsymbol{x}| = \sqrt{\langle \boldsymbol{x}, \boldsymbol{x} \rangle}$.

(1) For two elements $\boldsymbol{x}, \boldsymbol{y}$ in $\mathbb{C}^n$, show that $\langle A\boldsymbol{x}, \boldsymbol{y} \rangle = \langle \boldsymbol{x}, A^*\boldsymbol{y} \rangle$. Also, show that $\langle P\boldsymbol{x}, \boldsymbol{y} \rangle = \langle \boldsymbol{x}, P\boldsymbol{y} \rangle$.

(2) Let I be the $n \times n$ identity matrix. For any $\boldsymbol{x} \in \mathrm{Im}(P)$ and $\boldsymbol{y} \in \mathrm{Im}(I - P)$, show that $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = 0$. Also, show that $\mathbb{C}^n = \mathrm{Im}(P) \oplus \mathrm{Im}(I - P)$. Here, for an $n \times n$ complex matrix B, the set $\mathrm{Im}(B)$ is the image of the linear map $f_B : \mathbb{C}^n \to \mathbb{C}^n$ ($f_B(\boldsymbol{x}) = B\boldsymbol{x}$) determined by B.

(3) For $\boldsymbol{x} \in \mathrm{Im}(P)$, show that $|A\boldsymbol{x}| = |\boldsymbol{x}|$. Also, for $\boldsymbol{x} \in \mathrm{Im}(I - P)$, show that $A\boldsymbol{x} = \boldsymbol{0}$.

(4) Show that any eigenvalue λ of A satisfies $|\lambda| \leqslant 1$.

2 Let n be a positive integer and define the function g_n on $\mathbb{R}$ by

$$g_n(x) = \begin{cases} n & (x \in [0, 1/n]) \\ 0 & (x \notin [0, 1/n]). \end{cases}$$

Also, suppose that f is a continuous function on $\mathbb{R}$ and define the function f_n on $\mathbb{R}$ by

$$f_n(x) = \int_{-\infty}^{\infty} f(t)\, g_n(x - t)\, dt.$$

(1) Show that the function f_n is continuous for each n.

(2) Show that $\lim_{n \to \infty} f_n(x) = f(x)$ for each $x \in \mathbb{R}$.

(3) Determine whether the convergence verified in (2) is uniform or not. If it is, then give a proof. Otherwise, find a counterexample and show that it is indeed a counterexample.

3 Consider the quadratic polynomial $f(z) = z^2 - 2pz + 1$ and quartic polynomial $g(z) = z^4 - z^3 + z^2 - z + 1$, where p is a real constant.

(1) Express the remainder when $g(z)$ is divided by $f(z)$ in terms of p.

(2) When $p = \cos\dfrac{\pi}{5}$, show that a complex number α satisfying $f(\alpha) = 0$ also satisfies $g(\alpha) = 0$.

(3) Calculate the value of $\cos\dfrac{\pi}{5}$.

(4) When $p = \cos\dfrac{\pi}{5}$, express the value of the improper integral

$$\int_{-\infty}^{\infty} \frac{f(t)}{g(t)}\,dt$$

in terms of p without using the imaginary unit i.

4 Consider a map $f : X \to Y$, where X and Y are topological spaces. The map f is said to be continuous if, for any open set O in Y, its preimage $f^{-1}(O)$ is an open set in X. The map f is said to be closed if, for any closed set E in X, its image $f(E)$ is a closed set in Y. Furthermore, for subsets $A \subset X$ and $B \subset Y$, the set $\bar{A}$ is the closure of A in X while $\bar{B}$ is the closure of B in Y.

(1) Show that f is closed if and only if $\overline{f(A)} \subset f(\bar{A})$ for any subset $A \subset X$.

(2) Show that f is continuous if and only if $f^{-1}(F)$ is a closed set in X for any closed set F in Y.

(3) Show that f is continuous if and only if $f(\bar{A}) \subset \overline{f(A)}$ for any subset $A \subset X$.