# Entrance Examination for Master's Program
# Graduate School of Mathematics
# Nagoya University
# 2022 Admission

## Part 1 of 2

July 31, 2021, 9:00 ∼12:00

**Note:**

1. Please do not turn pages until told to do so.
2. The problem sheet consists of the cover page and 4 single-sided pages. After the exam has begun, please first confirm that the number of pages and their printing and order are correct. Please report any problem immediately.
3. There are a total of 4 problems labeled $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$, respectively. Please **answer all 4 problems.**
4. The answering sheet consists of 4 pages. Please **confirm the number of pages**, and please **do not remove the staple**.
5. Please write the answers to problems $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$ on pages $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$ of the answering sheet, respectively.
6. **Please write name and application number in the space provided on each of the** 4 **pages in the answering sheet.**
7. The back side of the 4 pages in the answering sheet may also be used. If used, please check the box at the lower right-hand corner on the front side.
8. If the answering sheet staple is torn, or if additional paper is needed for calculations, please notify the exam proctor.
9. After the exam has ended, please hand in the 4-page answering sheet. The problem sheet and any additional sheets used for calculations may be taken home.

**Notation:**

The symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of integers, rational numbers, real numbers, and complex numbers, respectively.

**1** Let $t$ be a real parameter. Consider the linear transformation $f$ of $\mathbb{R}^4$ given by the $4 \times 4$ real matrix

$$A = \begin{pmatrix} -1 & -1 & -3 & 2 \\ 0 & -1+2t & -1 & 1 \\ 1 & 0 & 2 & -1 \\ -2+t & -1-t & -5+t & 3 \end{pmatrix}.$$

Assuming that $\mathbb{R}^4$ is given the standard inner product, answer the following questions.

(1) Find an element $\boldsymbol{u}_1$ in $\mathbb{R}^4$ whose length equals 1 such that $f(\boldsymbol{u}_1) = \mathbf{0}$ for every $t$.

(2) Find the dimension $k$ of $\operatorname{Ker} f$. In addition, when $k \geq 2$, find $\boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$ such that $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k\}$ form an orthonormal basis of $\operatorname{Ker} f$.

(3) Find all $t$ such that the dimension of $\operatorname{Ker} f \cap \operatorname{Im} f$ equals 1 and give an element of $\operatorname{Ker} f \cap \operatorname{Im} f$ that is not $\mathbf{0}$ for each such $t$.

**2** Let $V$ be the vector space over $\mathbb{C}$ spanned by all $2 \times 2$ complex matrices with the basis $\{E_{11}, E_{12}, E_{21}, E_{22}\}$, where

$$E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

For $A \in V$, let $f_A : V \to V$ be the linear map defined by

$$f_A(X) = {}^tAXA \quad (X \in V),$$

where ${}^tA$ is the transpose of $A$. Let $B$ be the matrix given by

$$B = \begin{pmatrix} -1 & 1 \\ 2 & 0 \end{pmatrix}.$$

(1) Find the representation matrix of $f_B$ with respect to the basis $\{E_{11}, E_{12}, E_{21}, E_{22}\}$.

(2) Find all eigenvalues of $f_B$. Furthermore, for each eigenvalue, give a basis for the corresponding eigenspace.

(3) Assuming that $A$ is a real symmetric $2 \times 2$ matrix with distinct eigenvalues $\alpha$ and $\beta$, find all eigenvalues of $f_A$ and the dimensions of the corresponding eigenspaces.

**3** (1) Let $D$ be the region in $\{(x,y) \in \mathbb{R}^2 \mid x > 0,\ y > 0\}$ enclosed by the four curves $xy = 1$, $xy = 2$, $y = x$, $y = 3x$. Find the value of the integral

$$\iint_D \frac{y}{x}\, dxdy.$$

(2) Let $a, b \in \mathbb{R}$. Find necessary and sufficient conditions on $a, b$ so that the improper integral

$$\int_1^\infty x^a (\log x)^b\, dx$$

converges.

**4** Let $f(x, y)$ be a real function of class $C^2$ on $\mathbb{R}^2$ such that $f = 0$ and $(f_x)^2 + (f_y)^2 = 1$ on the unit circle $S = \{(\cos\theta, \sin\theta) \,|\, \theta \in \mathbb{R}\}$ and, on any half-line in $\mathbb{R}^2$ starting at the origin, $f$ is monotone increasing as we move away from the origin. Let $(a, b)$ be a point on $S$ and answer the following questions.

(1) Show that the two vectors $(f_x(a, b), f_y(a, b))$ and $(-b, a)$ are orthogonal.

(2) Show that $f_x(a, b) = a$ and $f_y(a, b) = b$.

(3) Show that the matrix

$$A = \begin{pmatrix} f_{xx}(a, b) & f_{xy}(a, b) \\ f_{yx}(a, b) & f_{yy}(a, b) \end{pmatrix}$$

satisfies $A \begin{pmatrix} -b \\ a \end{pmatrix} = \begin{pmatrix} -b \\ a \end{pmatrix}$.

(4) Calculate the limit

$$\lim_{n \to \infty} n^2 f\left(a - \frac{b}{n}, b + \frac{a}{n}\right).$$