Entrance Examination for Master's Program
Graduate School of Mathematics
Nagoya University
2021 Admission

Part 1 of 2

February 6, 2021, 10:00 〜12:00

Note:

1. Please do not turn pages until told to do so.
2. The problem sheet consists of the cover page and 3 single-sided pages. After the exam has begun, please first confirm that the number of pages and their printing and order are correct. Please report any problem immediately.
3. There are a total of 3 problems labeled **1** , **2** , and **3** respectively. Please **answer all 3 problems.**
4. The answering sheet consists of 3 single-sided pages. Please **confirm the number of pages**, and please **do not remove the staple**.
5. Please write the answers to problems **1** , **2** , and **3** on pages **1** , **2** , and **3** of the answering sheet, respectively.
6. **Please write name and application number in the space provided on each of the 3 pages in the answering sheet.**
7. The back side of the 3 pages in the answering sheet may also be used. If used, please check the box at the lower right-hand corner on the front side.
8. If the answering sheet staple is torn, or if additional paper is needed for calculations, please notify the exam proctor.
9. After the exam has ended, please hand in the 3 page answering sheet. The problem sheet and any additional sheets used for calculations may be taken home.

Notation:

The symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of integers, rational numbers, real numbers, and complex numbers, respectively.

1 Answer the following questions about the following 3×3 matrix where a and b are real numbers.

$$A = \begin{pmatrix} -b & b+1 & a \\ -2b & 2b+1 & a \\ -b-1 & b+1 & a+1 \end{pmatrix}$$

(1) Find all the eigenvalues of A for which there exists an eigenvector of the form $\begin{pmatrix} 1 \\ 1 \\ t \end{pmatrix}$ for some real number t.

(2) Find all the eigenvalues of A.

(3) For the case $b = 1$, calculate the dimension of the eigenspace corresponding to the eigenvalue 1.

(4) Find necessary and sufficient conditions on a, b so that A is not diagonalizable.

2 Answer the following questions.

(1) Prove that the improper integral

$$\int_1^\infty \frac{\cos x}{x}\,dx$$

converges.

(2) Find all the extremal values of the 2-variable function $f(x,y) = x^2 - y^4 - (x+y)^4$ defined on $\mathbb{R}^2$.

(3) Let $a > 0$. Find the value of the following integral.

$$\int_0^a \left(\int_y^{\sqrt{2a^2-y^2}} e^{x^2+y^2}\,dx \right) dy.$$

3 Consider the improper integral

$$I = \int_{-\infty}^{\infty} \frac{1}{x^6 + 1}\, dx.$$

Answer the following questions.

(1) Compute the poles of the complex function $f(z) = \dfrac{1}{z^6 + 1}$ in the upper half plane $\{z = x + iy \in \mathbb{C}\,;\ x, y \in \mathbb{R},\ y > 0\}$. Also compute the residues of f at these poles.

(2) Compute the value of I using the residue theorem.