Entrance Examination for Master's Program
Graduate School of Mathematics
Nagoya University
2020 Admission

Part 2 of 2

July 27, 2019, 13:00 ∼16:00

Note:

1. Please do not turn pages until told to do so.
2. The problem sheet consists of the cover page and 4 single-sided pages. After the exam has begun, please first confirm that the number of pages and their printing and order are correct. Please report any problem immediately.
3. There are a total of 4 problems labeled $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$, respectively. Please **answer all 4 problems.**
4. The answering sheet consists of 4 pages. Please **confirm the number of pages**, and please **do not remove the staple**.
5. Please write the answers to problems $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$ on pages $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$ of the answering sheet, respectively.
6. **Please write name and application number in the space provided on each of the 4 pages in the answering sheet.**
7. The back side of the 4 pages in the answering sheet may also be used. If used, please check the box at the lower right-hand corner on the front side.
8. If the answering sheet staple is torn, or if additional paper is needed for calculations, please notify the exam proctor.
9. After the exam has ended, please hand in the 4-page answering sheet. The problem sheet and any additional sheets used for calculations may be taken home.

Notation:

The symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of integers, rational numbers, real numbers, and complex numbers, respectively.

1 Let A, B be $n \times n$ complex matrices satisfying $A+B=I$, where I is the $n \times n$ identity matrix, and suppose that A is diagonalizable.

(1) Show that B is also diagonalizable.

(2) Show that, for an arbitrary eigenvalue λ of A, there exists an eigenvalue μ of B such that $\lambda + \mu = 1$.

(3) Show that, if the sum of the ranks of A and B equals n, then each of A and B has no eigenvalues other than $0, 1$.

2 Answer the following questions.

(1) For an arbitrary $x \in \mathbb{R}$, show that the improper integral

$$\int_0^\infty e^{-t^2} \cos(xt)\, dt$$

converges absolutely.

For (2)–(4), consider the function $f(x) = \displaystyle\int_0^\infty e^{-t^2} \cos(xt)\, dt$ defined on $\mathbb{R}$.

(2) Show that $f(x)$ is differentiable on $\mathbb{R}$.

(3) Show that $y = f(x)$ satisfies the ordinary differential equation $y' = -\dfrac{xy}{2}$.

(4) Compute the value of $f(2)$.

3 Let $t \in \mathbb{R}$ and consider the complex function

$$f(z) = \frac{e^{itz}}{(z^2+1)^2},$$

where i is the imaginary unit.

(1) Find all singularities of $f(z)$ in the complex plane $\mathbb{C}$.

(2) For each of the isolated singularities in $\mathbb{C}$ with positive imaginary part, find the principal part (the part with negative powers) of the Laurent series of $f(z)$ at that point.

(3) Compute the improper integral

$$\int_{-\infty}^{\infty} \frac{e^{itx}}{(x^2+1)^2}\, dx.$$

4 Suppose that a point sequence $a(k)$ $(k = 1, 2, \ldots)$ in the n-dimensional Euclidean space $\mathbb{R}^n$ converges to $a \in \mathbb{R}^n$. Here, consider the standard topology on $\mathbb{R}^n$, that is, the topology that arises by defining the distance $d(x, y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$ between two points $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. For the subset $A = \{a(k)\,; k = 1, 2, \ldots, \}$ of $\mathbb{R}^n$, answer the following questions.

(1) Find the closure $\overline{A}$ of A in $\mathbb{R}^n$.

(2) For a closed subset B of $\mathbb{R}^n$ satisfying $\overline{A} \cap B = \emptyset$, determine whether

$$\inf\{d(x, y)\,; x \in A,\ y \in B\} > 0$$

holds. If it does, then give a proof. Otherwise, give a counterexample and show that it is indeed a counterexample.