Entrance Examination for Master's Program
Graduate School of Mathematics
Nagoya University
2018 Admission

Part 2 of 2

February 6, 2018, 13:00 ∼16:00

Note:

1. Please do not turn pages until told to do so.
2. The problem sheet consists of the cover page and 4 single-sided pages. After the exam has begun, please first confirm that the number of pages and their printing and order are correct. Please report any problem immediately.
3. There are a total of 4 problems labeled $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$, respectively. Please **answer all 4 problems.**
4. The answering sheet consists of 4 single-sided pages. Please **confirm the number of pages**, and please **do not remove the staple**.
5. Please write the answers to problems $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$ on pages $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$ of the answering sheet, respectively.
6. **Please write name and application number in the space provided on each of the 4 pages in the answering sheet.**
7. The back side of the 4 pages in the answering sheet may also be used. If used, please check the box at the lower right-hand corner on the front side.
8. If the answering sheet staple is torn, or if additional paper is needed for calculations, please notify the exam proctor.
9. After the exam has ended, please hand in the 4 page answering sheet. The problem sheet and any additional sheets used for calculations may be taken home.

Notation:

The symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of integers, rational numbers, real numbers, and complex numbers, respectively.

1 Let A and ${}^tA$ be an $n \times n$ real matrix and its transpose, respectively. Prove the following statements.

(1) If $u_1, u_2 \in \mathbb{R}^n$ are orthogonal and both are eigenvectors of ${}^tAA$, then Au_1, Au_2 are orthogonal.

(2) If A is invertible, then there exist $n \times n$ real orthogonal matrices U, V and a diagonal real matrix D whose entries are nonnegative such that $AU = VD$.

(3) If A is invertible, then there exist an $n \times n$ real orthogonal matrix T and an $n \times n$ real symmetric matrix R whose eigenvalues are all nonnegative such that $R^2 = {}^tAA$, $A = TR$.

(4) Prove that both (2) and (3) hold even if A is not invertible.

Note: If you give correct proofs that do not depend on the fact that A is invertible for both (2) and (3), then it is assumed that (4) is already answered. If that is the case, then there is no need of working on (4) again.

2 For a function $f : \mathbb{R} \to \mathbb{R}$, assume that the right limit $\ell_+(x) = \lim_{\substack{y \to x \\ y > x}} f(y)$ and the left limit $\ell_-(x) = \lim_{\substack{y \to x \\ y < x}} f(y)$ both exist and are finite for all $x \in \mathbb{R}$.

(1) Using the ε-δ definition of limit, state precisely that the real number $\ell_+(x)$ equals the right-limit $\lim_{\substack{y \to x \\ y > x}} f(y)$.

(2) Show that, for every $x \in \mathbb{R}$, there exists $\delta > 0$ such that f is bounded on the interval $(x - \delta, x + \delta)$.

(3) Show that f is bounded on an arbitrary bounded interval.

3 For $L > 0$, let C be the contour along the circumference of the triangle in the complex plane whose vertices are at 0, L, $L + Li$, traversed counterclockwise. By using the result of integrating $f(z) = \exp(-z^2/2)$ $(z \in \mathbb{C})$ along C, find the value of the following improper integral:

$$\int_0^\infty \exp(-ix^2)\,dx.$$

You may use the fact that $\int_0^\infty \exp(-x^2/2)\,dx = \sqrt{\pi/2}$ without proof.

4 Let (X, d) be a metric space with a metric d. For nonempty $A, B \subset X$, let $d(A, B)$ be defined as follows:

$$d(A, B) = \inf\{d(a, b)\ ;\ a \in A,\ b \in B\}.$$

Also, for $x \in X$, define $d(x, B) = d(\{x\}, B)$.

(1) For nonempty $A, B \subset X$, show the following:

$$d(A, B) = \inf\{d(a, B)\ ;\ a \in A\}.$$

(2) For $x, y \in X$ and nonempty $B \subset X$, show the following:

$$|d(x, B) - d(y, B)| \leq d(x, y).$$

(3) Determine whether or not the following statement holds. If it does, then prove it. Otherwise, find a counterexample and show that it is in fact a counterexample.

> If $A, B \subset X$ are both nonempty closed sets and $A \cap B = \emptyset$, then $d(A, B) > 0$.

(4) Determine whether or not the following statement holds. If it does, then prove it. Otherwise, find a counterexample and show that it is in fact a counterexample.

> If $A \subset X$ is a nonempty compact set, $B \subset X$ is a nonempty closed set, and $A \cap B = \emptyset$, then $d(A, B) > 0$.