Entrance Examination for Master's Program
Graduate School of Mathematics
Nagoya University
2018 Admission

Part 1 of 2

July 29, 2017, 9:00 ∼12:00

Note:

1. Please do not turn pages until told to do so.
2. The problem sheet consists of the cover page and 4 single-sided pages. After the exam has begun, please first confirm that the number of pages and their printing and order are correct. Please report any problem immediately.
3. There are a total of 4 problems labeled $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$, respectively. Please **answer all 4 problems.**
4. The answering sheet consists of 4 pages. Please **confirm the number of pages**, and please **do not remove the staple**.
5. Please write the answers to problems $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$ on pages $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$ of the answering sheet, respectively.
6. **Please write name and application number in the space provided on each of the 4 pages in the answering sheet.**
7. The back side of the 4 pages in the answering sheet may also be used. If used, please check the box at the lower right-hand corner on the front side.
8. If the answering sheet staple is torn, or if additional paper is needed for calculations, please notify the exam proctor.
9. After the exam has ended, please hand in the 4-page answering sheet. The problem sheet and any additional sheets used for calculations may be taken home.

Notation:

The symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of integers, rational numbers, real numbers, and complex numbers, respectively.

1 Let V be a real vector space whose elements are polynomials of degree at most three of real variable x with real coefficients. For real numbers p, q, r, we define a linear map $T : V \to V$ by

$$T(f(x)) = pf(x) + (qx + r)f'(x).$$

Answer the following questions.

(1) Find the representation matrix of T with respect to the basis $\{1, x, x^2, x^3\}$ of V.

(2) Find the necessary and sufficient condition for p, q, r so that the dimension of $\operatorname{Ker} T$ is 0.

(3) Find the necessary and sufficient condition for p, q, r so that the dimension of $\operatorname{Im} T$ is 3.

2 Let $M_2(\mathbb{C})$ be the set of all complex square matrices of degree two. Also, let $GL_2(\mathbb{C})$ be the set of all the regular matrices in the set $M_2(\mathbb{C})$. Further, for $A \in M_2(\mathbb{C})$, we define $Z(A)$ as the set of all the elements $X \in M_2(\mathbb{C})$ such that $AX = XA$. Prove the following statements.

(1) When $A, X \in M_2(\mathbb{C})$, $P \in GL_2(\mathbb{C})$,

$$X \in Z(A) \iff P^{-1}XP \in Z(P^{-1}AP).$$

(2) If $A \in GL_2(\mathbb{C})$ has two different eigenvalues, then, there exists $P \in GL_2(\mathbb{C})$ such that

$$Z(A) = \left\{ P \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} P^{-1} \ ; \ x, y \in \mathbb{C} \right\}.$$

(3) If $A \in GL_2(\mathbb{C})$ is not diagonalizable, then, there exists $P \in GL_2(\mathbb{C})$ such that

$$Z(A) = \left\{ P \begin{pmatrix} x & y \\ 0 & x \end{pmatrix} P^{-1} \ ; \ x, y \in \mathbb{C} \right\}.$$

3 Answer the following questions. (They are independent questions.)

(1) Let $\cos^{-1} : (-1, 1) \to (0, \pi)$ be the inverse function of $\cos : (0, \pi) \to (-1, 1)$. We define $f(x) = \cos(a \cos^{-1} x)$ $(x \in (-1, 1))$ with a real number a. Show that $(1 - x^2)f''(x)$ for $x \in (-1, 1)$ is a linear combination of $f(x)$ and $xf'(x)$, and compute the coefficients.

(2) Let $p, q \geq 0$. We define

$$f(x, y) = \frac{|x|^p |y|^q}{x^2 + y^2}$$

for $(x, y) \in \mathbb{R}^2$, $(x, y) \neq (0, 0)$. Find the necessary and sufficient condition for p, q so that, by appropriately setting the value of $f(0, 0)$, $f : \mathbb{R}^2 \to \mathbb{R}$ is totally differentiable at the origin.

(3) Let a and ϵ be positive real numbers. We set

$$D_\epsilon = \{(x, y) \,|\, \epsilon \leq x \leq 1,\ x^a \leq y \leq 1\}.$$

Compute the following limit

$$\lim_{\epsilon \to 0} \iint_{D_\epsilon} \log y \, dx dy.$$

4 Let $f : [0, \infty) \to [0, \infty)$ be a continuous and monotone non-increasing function. Prove the following statements.

(1) If $\displaystyle\sum_{n=0}^{\infty} f(na) < \infty$ for some $a > 0$, then, $\displaystyle\int_0^{\infty} f(x)dx < \infty$.

(2) If $\displaystyle\int_0^{\infty} f(x)|\sin x|dx < \infty$, then, $\displaystyle\int_0^{\infty} f(x)dx < \infty$.