Entrance Examination for Master's Program
Graduate School of Mathematics
Nagoya University
2017 Admission

Part 1 of 2

February 7, 2017, 9:00 ∼12:00

Note:

1. Please do not turn pages until told to do so.
2. The problem sheet consists of the cover page and 4 single-sided pages. After the exam has begun, please first confirm that the number of pages and their printing and order are correct. Please report any problem immediately.
3. There are a total of 4 problems labeled $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$, respectively. Please **answer all 4 problems.**
4. The answering sheet consists of 4 single-sided pages. Please **confirm the number of pages**, and please **do not remove the staple**.
5. Please write the answers to problems $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$ on pages $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$ of the answering sheet, respectively.
6. **Please write name and application number in the space provided on each of the** 4 **pages in the answering sheet.**
7. The back side of the 4 pages in the answering sheet may also be used. If used, please check the box at the lower right-hand corner on the front side.
8. If the answering sheet staple is torn, or if additional paper is needed for calculations, please notify the exam proctor.
9. After the exam has ended, please hand in the 4 page answering sheet. The problem sheet and any additional sheets used for calculations may be taken home.

Notation:

The symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of integers, rational numbers, real numbers, and complex numbers, respectively.

1 Let V be the real linear space consisting of polynomials in x of degree at most 3 with real coefficients. Given a real number α, for $f(x) \in V$ we define

$$\Psi(f(x)) = \frac{f(x) - f(\alpha)}{x - 1}.$$

(1) Find a necessary and sufficient condition on α so that $\Psi(f(x)) \in V$ for any $f(x) \in V$. Show that Ψ is a linear map from V to V, when this condition is satisfied.

Below, we assume that the condition found in (1) is satisfied. Given a real number β, consider the linear map Φ from V to V defined by

$$\Phi(f(x)) = \frac{f(x) - f(\alpha)}{x - 1} + \beta\big(f'(x) - f'(0)\big) - \frac{\beta}{6}\big(f''(x) - f''(1)\big)(x - 1).$$

Here $f'(x)$ and $f''(x)$ are the first and second derivatives of $f(x)$ respectively.

(2) Find the representation matrix of Φ with respect to the basis $\{1, (x-1), (x-1)^2, (x-1)^3\}$ of V.

(3) Find the dimension of the kernel of Φ.

2 Given a complex number a, consider the following 3×3 complex matrix

$$A = \begin{pmatrix} -a & a+2 & -3a-4 \\ -a+1 & a+1 & -3a+2 \\ 1 & -1 & 4 \end{pmatrix}.$$

(1) Find the eigenvalues and eigenspaces of A.

(2) Find the minimal polynomial of A.

(3) Assume that $a = 0$. Let

$$\vec{x} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

be a vector in the three dimensional complex linear space $\mathbb{C}^3$. Find a necessary and sufficient condition on p, q, r such that there exists a vector $\vec{y}$ in $\mathbb{C}^3$ satisfying

$$\vec{x} = (A - 2E)\vec{y}.$$

Here E is the identity matrix.

3 Each of (1), (2), (3) is an independent problem.

(1) Let $f(x, y)$ be a differentiable function on $\mathbb{R}^2$, and $(a, b) \in \mathbb{R}^2$ be a given point. Suppose that

$$a\frac{\partial f}{\partial x}(ta, tb) + b\frac{\partial f}{\partial y}(ta, tb) \geq 0$$

for any $t \in (0, 1)$. Then show that $f(a, b) \geq f(0, 0)$.

(2) Let α be a real number. Determine the necessary and sufficient condition on α so that the following improper integral converges.

$$\int_0^\infty \frac{1 - \cos x}{x^\alpha}\, dx.$$

(3) Let $f_0(x)$ be a continuous function on $[a, b]$. We define functions $f_1(x)$, $f_2(x)$, $f_3(x), \ldots$ inductively by

$$f_n(x) = \int_a^x f_{n-1}(y)\, dy, \quad x \in [a, b], \quad n = 1, 2, 3, \ldots$$

(i) Show that $f_2(x) = \displaystyle\int_a^x f_0(z)(x - z)dz$.

(ii) Show that

$$|f_n(x)| \leq \frac{(b-a)^n}{n!} \max_{a \leq x \leq b} |f_0(x)|$$

for any $n = 1, 2, 3, \ldots$

4 Let φ be a function of class C^1 defined on $\mathbb{R}$. Suppose that $\varphi(0) > 0$, and that $\varphi(x) = 0$ whenever $|x| > 1$.

(1) Let $f(x,t) = \varphi(x-t)$. Compute $\dfrac{\partial f}{\partial t} + \dfrac{\partial f}{\partial x}$.

(2) Let the sequence of functions $\{g_n\}_{n=1}^{\infty}$ be defined by $g_n(x) = \varphi(x-n)$. Find the pointwise limit g of $\{g_n\}_{n=1}^{\infty}$. Does $\{g_n\}_{n=1}^{\infty}$ converge uniformly to g on $\mathbb{R}$? Explain the reason for your answer.

Here, we say that a sequence of functions $\{g_n\}_{n=1}^{\infty}$ converges pointwise to g on $\mathbb{R}$ if

$$\lim_{n\to\infty} g_n(x) = g(x)$$

for any $x \in \mathbb{R}$. We say that a sequence of functions $\{g_n\}_{n=1}^{\infty}$ converges uniformly to g on $\mathbb{R}$ if

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} |g_n(x) - g(x)| = 0.$$

(3) Let the sequence of functions $\{h_n\}_{n=1}^{\infty}$ be defined by $h_n(x) = \varphi(x - \frac{1}{n})$. Find the pointwise limit h of $\{h_n\}_{n=1}^{\infty}$. Does $\{h_n\}_{n=1}^{\infty}$ converge uniformly to h on $\mathbb{R}$? Explain the reason for your answer.