Entrance Examination for Master's Program
Graduate School of Mathematics
Nagoya University
2016 Admission

Part 2 of 2

Saturday, July 25, 2015, 13:00 p.m.∼16:00 p.m.

Note:

1. Please do not turn pages until told to do so.
2. The problem sheet consists of the cover page and 4 single-sided pages. After the exam has begun, please first confirm that the number of pages and their printing and order are correct. Please report any problem immediately.
3. There are a total of 4 problems labeled [1], [2], [3], and [4], respectively. Please **answer all 4 problems.**
4. The answering sheet consists of 4 pages. Please **confirm the number of pages**, and please **do not remove the staple**.
5. Please write the answers to problems [1], [2], [3], and [4] on pages [1], [2], [3], and [4] of the answering sheet, respectively.
6. **Please write name and application number in the space provided on each of the 4 pages in the answering sheet.**
7. The back side of the 4 pages in the answering sheet may also be used. If used, please check the box at the lower right-hand corner on the front side.
8. If the answering sheet staple is torn, or if additional paper is needed for calculations, please notify the exam proctor.
9. After the exam has ended, please hand in the 4-page answering sheet. The problem sheet and any additional sheets used for calculations may be taken home.

Notation:

The symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of integers, rational numbers, real numbers, and complex numbers, respectively.

1 Let V, W be finite-dimensional vector spaces over $\mathbb{C}$. The dual space V^* of V is the vector space over $\mathbb{C}$ consisting of all linear maps from V to $\mathbb{C}$. Also, for a linear map $f : V \to W$, let $f^* : W^* \to V^*$ be the linear map defined by

$$f^*(h) = h \circ f, \quad h \in W^*.$$

(1) For a basis $e_1, \ldots, e_n$ of V, suppose that the vectors $e_1^*, \ldots, e_n^* \in V^*$ satisfy the condition:

$$\text{For any } i, j \in \{1, \ldots, n\},\ e_i^*(e_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases}$$

Show that $e_1^*, \ldots, e_n^*$ is a basis of V^*.

(2) Show that if $f : V \to W$ is surjective, then f^* is injective.

(3) Show that if $f : V \to W$ is injective, then f^* is surjective.

2 For two points (x, y), (x', y') in $\mathbb{R}^2$, define

$$(x, y) - (x', y') = (x - x', y - y'), \quad \|(x, y)\| = \sqrt{x^2 + y^2}.$$

In the following, properties of continuous functions on a bounded closed set in $\mathbb{R}^2$ may be used without proofs. In such a case, be sure to clearly state what properties are used.

(1) Suppose that a real function $f(x, y)$ is continuous on $\mathbb{R}^2$, $f(x_0, y_0) > 0$ for some point (x_0, y_0), and $\displaystyle\lim_{\|(x,y)\|\to\infty} f(x, y) = 0$. Show that $f(x, y)$ attains a maximum at some point in $\mathbb{R}^2$.

(2) Suppose that a real function $g(x, y)$ is of class C^1 on $\mathbb{R}^2$ and satisfies

$$\lim_{\|(x,y)\|\to\infty} \frac{\partial g}{\partial x}(x, y) = 0, \qquad \lim_{\|(x,y)\|\to\infty} \frac{\partial g}{\partial y}(x, y) = 0.$$

Show that there exists a constant $L > 0$ such that

$$|g(x, y) - g(x', y')| \leq L\|(x, y) - (x', y')\|$$

for any (x, y), (x', y') in $\mathbb{R}^2$.

3 Consider the complex function $f(z) = \dfrac{e^{az}}{1+e^{z}}$, where a is a constant with $0 < a < 1$.

(1) Find all singularities of $f(z)$ in $\mathbb{C}$.

(2) For a real number $R > 0$, let Γ_R be the contour along the circumference of the rectangle in the complex plane whose corners are at R, $R+2\pi i$, $-R+2\pi i$, $-R$, traversed counterclockwise. Find the value of the integral

$$\int_{\Gamma_R} f(z)\,dz.$$

(3) Find the value of the improper integral $I = \displaystyle\int_{-\infty}^{\infty} f(x)\,dx$.

4 Let (X, d) be a metric space with a metric d. For $a \in X$ and a positive real number r, the open ball of radius r centered at a is the set

$$B_r(a) = \{x \in X \mid d(a, x) < r\}.$$

(1) Using open balls, define that a subset U of X is an open set.

(2) Using open balls, define that a sequence $\{x_n\}_{n=1}^{\infty}$ of points in the metric space (X, d) converges to $x_\infty \in X$.

A subset K of a topological space X is said to be compact in X if the following holds:

For an arbitrary family $\{U_\lambda \mid \lambda \in \Lambda\}$ of open sets in X such that $K \subset \bigcup_{\lambda \in \Lambda} U_\lambda$, there exists a finite subset Λ_0 of Λ for which $K \subset \bigcup_{\lambda \in \Lambda_0} U_\lambda$.

Answer (3) and (4) based on this definition.

(3) Suppose that a sequence $\{x_n\}_{n=1}^{\infty}$ of points in the metric space (X, d) converges to $x_\infty \in X$. Let

$$A = \{x_n \mid n = 1, 2, \dots\} \cup \{x_\infty\}.$$

If A is compact in X, prove it. Otherwise, find a counterexample and show that it is in fact a counterexample.

(4) Suppose that a sequence $\{x_n\}_{n=1}^{\infty}$ of points in the metric space (X, d) converges to $x_\infty \in X$. Let

$$B = \{x_n \mid n = 1, 2, \dots\}.$$

If B is compact in X, prove it. Otherwise, find a counterexample and show that it is in fact a counterexample.