Entrance Examination for Master's Program
Graduate School of Mathematics
Nagoya University
2016 Admission

Part 1 of 2

Saturday, July 25, 2015, 9:00 a.m.～12:00 noon

Note:

1. Please do not turn pages until told to do so.
2. The problem sheet consists of the cover page and 4 single-sided pages. After the exam has begun, please first confirm that the number of pages and their printing and order are correct. Please report any problem immediately.
3. There are a total of 4 problems labeled $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$, respectively. Please **answer all 4 problems.**
4. The answering sheet consists of 4 pages. Please **confirm the number of pages**, and please **do not remove the staple**.
5. Please write the answers to problems $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$ on pages $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$ of the answering sheet, respectively.
6. **Please write name and application number in the space provided on each of the 4 pages in the answering sheet.**
7. The back side of the 4 pages in the answering sheet may also be used. If used, please check the box at the lower right-hand corner on the front side.
8. If the answering sheet staple is torn, or if additional paper is needed for calculations, please notify the exam proctor.
9. After the exam has ended, please hand in the 4-page answering sheet. The problem sheet and any additional sheets used for calculations may be taken home.

Notation:

The symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of integers, rational numbers, real numbers, and complex numbers, respectively.

1 (1) Let p, q, r be real numbers. Find

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4$$

satisfying the system

$$\begin{cases} x_1 - 2x_2 - x_4 & = 1, \\ 2x_1 - 4x_2 - (r^2 + r - 2)x_3 - 2x_4 & = 3p - 3q + 2, \\ (2 - 3p)x_1 - (4 - 6p)x_2 + (3p + r)x_4 & = 2. \end{cases}$$

(2) Let s be a real number and consider the subspace V of $\mathbb{R}^4$ spanned by the three vectors

$$\vec{a} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \ \vec{b} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \ \vec{c} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ s \end{pmatrix}.$$

Furthermore, let W be the set of vectors $\vec{x}$ in (1) corresponding to $q = r = 0$ and $p \neq 0$. Find s for which $V \cap W \neq \emptyset$.

2 Let P_n be the $(n+1)$-dimensional vector space over $\mathbb{C}$ consisting of all polynomials in x of degree at most n with coefficients in $\mathbb{C}$. Let $f_n : P_n \to P_n$ be the linear transformation given by

$$f_n : p(x) \mapsto p(x) + p''(x) + p'''(x), \quad p(x) \in P_n,$$

where $p''(x)$ and $p'''(x)$ are the second and third derivatives of $p(x)$, respectively.

(1) Find the representation matrix of f_4 with respect to the basis x^4, x^3, x^2, x, 1 of P_4.

(2) Find all eigenvalues of f_4 and the corresponding eigenspaces.

(3) Find the minimal polynomial of f_4 as well as a Jordan canonical form of a representation matrix of f_4. (Here, there is no need to find an invertible matrix that transforms a representation matrix into a Jordan canonical form. Assume the same for (4).)

(4) For each $n \geq 1$, find a Jordan canonical form of a representation matrix of f_n.

3 (1) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an orthogonal matrix with real entries. Suppose that $u(s,t)$ is a real function on $\mathbb{R}^2$ that is of class C^2 and satisfies $\dfrac{\partial^2 u}{\partial s^2} + \dfrac{\partial^2 u}{\partial t^2} = 0$. Show that, if

$$v(x,y) = u(ax+by, cx+dy), \quad (x,y) \in \mathbb{R}^2,$$

then $v(x,y)$ satisfies $\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = 0$.

(2) For the real function

$$f(x) = \int_0^x e^{e^{t+x^2}}\, dt,$$

find the value of the derivative $f'(1)$.

(3) For a real number $a > 0$ and an integer $k \geq 1$, show that the sequence

$$c_n = \frac{(1+a)^n}{n^k}, \quad n = 1, 2, \ldots$$

diverges as $n \to \infty$.

4 Let V be the subset of $\mathbb{R}^3$ given by

$$V = \{(x, y, z) \in \mathbb{R}^3 \mid z > 0,\ \sqrt{x^2 + y^2} \leq -z \log z\}.$$

(1) Let $c \in \mathbb{R}$. Find the necessary and sufficient condition on c so that the intersection of the plane $z = c$ and V is nonempty.

(2) Let a, $b \in \mathbb{R}$. Find the necessary and sufficient condition on a and b so that the improper integral

$$\int_0^\infty e^{ax} x^b dx$$

converges.

(3) Let $\alpha \in \mathbb{R}$. Find the necessary and sufficient condition on α so that the improper triple integral

$$\iiint_V \frac{(x^2 + y^2)^\alpha}{z^2}\, dxdydz$$

converges.