Entrance Examination for the Ph. D. Program
Graduate School of Mathematics
Nagoya University
2015 Admission

Part 2 of 2

Thursday, February 5, 2015, 13:00 p.m.∼16:00 p.m.

Note:

1. Please do not turn pages until told to do so.
2. The problem sheet consists of the cover page and 4 single-sided pages. After the exam has begun, please first confirm that the number of pages and their printing and order are correct. Please report any problem immediately.
3. There are a total of 4 problems labeled $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$, respectively. Please **answer all 4 problems.**
4. The answering sheet consists of 4 single-sided pages. Please **confirm the number of pages**, and please **do not remove the staple**.
5. Please write the answers to problems $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$ on pages $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$ of the answering sheet, respectively.
6. **Please write name and application number in the space provided on each of the 4 pages in the answering sheet.**
7. The back side of the 4 pages in the answering sheet may also be used. If used, please check the box at the lower right-hand corner on the front side.
8. If the answering sheet staple is torn, or if additional paper is needed for calculations, please notify the exam proctor.
9. After the exam has ended, please hand in the 4 page answering sheet. The problem sheet and any additional sheets used for calculations may be taken home.

Notation:

The symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of integers, rational numbers, real numbers, and complex numbers, respectively.

1 Let V be a vectorspace over the field K and W a subvectorspace of V. Define the equivalence relation $\sim$ by

$$v_1 \sim v_2 \quad \Longleftrightarrow \quad v_1 - v_2 \in W$$

on V, and let V/W be the quotient set with respect to the equivalence relation $\sim$. If we denote the element of V/W represented by $v \in V$ by $\overline{v}$, and define addition and scalar multiplication in V/W by

$$\overline{v_1} + \overline{v_2} = \overline{v_1 + v_2}, \quad c\overline{v} = \overline{cv}, \quad (v_1, v_2, v \in V,\, c \in K),$$

then V/W is a vector space over K.

(1) Verify that the addition and scalar multiplication of V/W is well-defined (i.e., does not depend on the choice of the representative).

(2) Suppose $W \neq V$ and $W \neq \{0\}$, where 0 denotes the zero vector. Show that for the basis $v_1, \ldots, v_k$ of W there exits $v_{k+1}, \ldots, v_n \in V$ such that $v_1, \ldots, v_n$ is a basis of V.

(3) Use (2) to show that $\overline{v_{k+1}}, \ldots, \overline{v_n}$ is a basis of V/W.

2 Let $f(x_1, x_2, x_3)$ be a C^1-function which is defined in a neighborhood of $S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \,|\, x_1^2 + x_2^2 + x_3^2 = 1\}$ in $\mathbb{R}^3$. Define the function $g(u_1, u_2, u_3)$ on $\mathbb{R}^3$ minus the origin by

$$g(u_1, u_2, u_3) = f\left(\frac{u_1}{\sqrt{u_1^2 + u_2^2 + u_3^2}}, \frac{u_2}{\sqrt{u_1^2 + u_2^2 + u_3^2}}, \frac{u_3}{\sqrt{u_1^2 + u_2^2 + u_3^2}}\right).$$

Let $(\omega_1, \omega_2, \omega_3)$ be a fixed point of S^2.

(1) Show that the following inequality holds:

$$\sum_{j=1}^{3} \left(\frac{\partial g}{\partial u_j}(\omega_1, \omega_2, \omega_3)\right)^2 \leq \sum_{j=1}^{3} \left(\frac{\partial f}{\partial x_j}(\omega_1, \omega_2, \omega_3)\right)^2. \qquad (*)$$

(2) Consider the one-variable function

$$F(r) = f(r\omega_1, r\omega_2, r\omega_3)$$

defined in a neighborhood of 1. Show that equality holds in $(*)$ if and only if $F'(1) = 0$.

3 Consider the following complex function $f(z) = h - \cos z$, where h is a non-negative real number, and $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$.

(1) Find all zeroes of $f(z)$.

(2) For every zero of $f(z)$, find the residue of $\dfrac{1}{f(z)}$.

4 Let X and Y be topological spaces. Let X_1, X_2 be closed subsets of X satisfying $X = X_1 \cup X_2$. Consider continuous functions $f_1 \colon X_1 \longrightarrow Y$ and $f_2 \colon X_2 \longrightarrow Y$ such that $f_1(x) = f_2(x)$ for all $x \in X_1 \cap X_2$, where the topology of X_1 and X_2 is the relative toplogy with respect to X. Define the function $F \colon X \longrightarrow Y$ by

$$F(x) = \begin{cases} f_1(x) & (\text{for } x \in X_1) \\ f_2(x) & (\text{for } x \in X_2). \end{cases}$$

(1) Show that $F^{-1}(V) = f_1^{-1}(V) \cup f_2^{-1}(V)$ for any subset V of Y.

(2) Show that $F \colon X \longrightarrow Y$ is a continuous function.