Entrance Examination for the Ph. D. Program
Graduate School of Mathematics
Nagoya University
2015 Admission

Part 1 of 2

Thursday, February 5, 2015, 9:00 a.m.∼12:00 noon

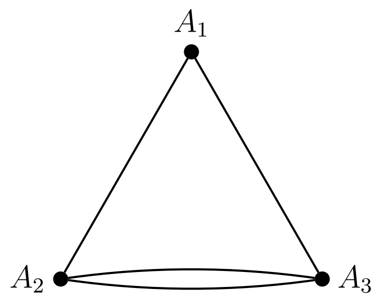
Note:

1. Please do not turn pages until told to do so.
2. The problem sheet consists of the cover page and 4 single-sided pages. After the exam has begun, please first confirm that the number of pages and their printing and order are correct. Please report any problem immediately.
3. There are a total of 4 problems labeled $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$, respectively. Please **answer all 4 problems.**
4. The answering sheet consists of 4 single-sided pages. Please **confirm the number of pages**, and please **do not remove the staple**.
5. Please write the answers to problems $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$ on pages $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$ of the answering sheet, respectively.
6. **Please write name and application number in the space provided on each of the 4 pages in the answering sheet.**
7. The back side of the 4 pages in the answering sheet may also be used. If used, please check the box at the lower right-hand corner on the front side.
8. If the answering sheet staple is torn, or if additional paper is needed for calculations, please notify the exam proctor.
9. After the exam has ended, please hand in the 4 page answering sheet. The problem sheet and any additional sheets used for calculations may be taken home.

Notation:

The symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of integers, rational numbers, real numbers, and complex numbers, respectively.

1 Consider the following graph with vertices A_1, A_2, A_3 and edges as indicated. Let m_i be the number of edges with endpoint A_i, k_{ij} the number of edges connecting A_i and A_j, and set $p_{ij} = k_{ij}/m_i$, Consider the 3×3-matrix $P = (p_{ij})$.



(1) Determine the matrix P, all of its eigenvalues, and the eigenspaces corresponding to these eigenvalues.

(2) For an integer n and $x_0 = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$, let $x_n = P^n x_0$. Find the limit $\lim_{n\to\infty} x_n$.

2 Let V be an $\mathbb{R}$-vector space and $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, $\mathbf{x}_4$ be linearly independent vectors in V. For given $a, b \in \mathbb{R}$, consider the following vectors

$$\mathbf{y}_1 = 2\,\mathbf{x}_2 + 4\,\mathbf{x}_3 + 2\,\mathbf{x}_4, \qquad \mathbf{y}_2 = 2\,\mathbf{x}_1 + a\,\mathbf{x}_2 - 4\,\mathbf{x}_3 + \mathbf{x}_4$$

$$\mathbf{y}_3 = -\,\mathbf{x}_1 + 2\,\mathbf{x}_2 + b\,\mathbf{x}_3 + 2\,\mathbf{x}_4, \quad \mathbf{y}_4 = 2\,\mathbf{x}_1 + \mathbf{x}_2 + 4\,\mathbf{x}_3 + 5\,\mathbf{x}_4.$$

(1) Determine for which values of a and b the vectors $\mathbf{y}_1$, $\mathbf{y}_2$, $\mathbf{y}_3$, $\mathbf{y}_4$ are linearly independent.

(2) Determine for which values of a and b the vectors $\mathbf{y}_1$, $\mathbf{y}_2$, $\mathbf{y}_3$ are linearly dependent.

3 For $t > 0$, and $x, y \in \mathbb{R}$, define

$$u(t,x,y) = \frac{1}{t} \exp\left(-\frac{x^2+y^2}{t}\right).$$

(1) Determine for which x and y the limit

$$\lim_{t \to +0} u(t,x,y)$$

exists, and determine its value.

(2) Find the value of the double integral

$$\iint_{\mathbb{R}^2} u(t,x,y)\, dxdy.$$

(3) Find the constant k for which the equation

$$k \frac{\partial u}{\partial t}(t,x,y) = \frac{\partial^2 u}{\partial x^2}(t,x,y) + \frac{\partial^2 u}{\partial y^2}(t,x,y)$$

holds for all $t > 0$, x, $y \in \mathbb{R}$.

4 Consider the plane curve defined by the equation

$$C = \{(x, y) \in \mathbb{R}^2 \mid x^3 - y^3 + x - y - 4 = 0\}.$$

(1) Show that the maximum of the distance of points (x, y) on C to the origin $(0, 0)$ does not exist.

(2) Find the minimum of the distance of points (x, y) on C to the origin $(0, 0)$.