Entrance Examination for the Ph. D. Program
Graduate School of Mathematics
Nagoya University
2015 Admission

Part 1 of 2

Saturday, July 26, 2014, 9:00 a.m.∼12:00 noon

Note:

1. Please do not turn pages until told to do so.
2. The problem sheet consists of the cover page and 4 single-sided pages. After the exam has begun, please first confirm that the number of pages and their printing and order are correct. Please report any problem immediately.
3. There are a total of 4 problems labeled [1], [2], [3], and [4], respectively. Please **answer all 4 problems.**
4. The answering sheet consists of 4 single-sided pages. Please **confirm the number of pages**, and please **do not remove the staple**.
5. Please write the answers to problems [1], [2], [3], and [4] on pages [1], [2], [3], and [4] of the answering sheet, respectively.
6. **Please write name and application number in the space provided on each of the** 4 **pages in the answering sheet.**
7. The back side of the 4 pages in the answering sheet may also be used. If used, please check the box at the lower right-hand corner on the front side.
8. If the answering sheet staple is torn, or if additional paper is needed for calculations, please notify the exam proctor.
9. After the exam has ended, please hand in the 4-page answering sheet. The problem sheet and any additional sheets used for calculations may be taken home.

Notation:

The symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of integers, rational numbers, real numbers, and complex numbers, respectively.

1 Let V be the vector space over $\mathbb{C}$ consisting of polynomials in x of degree at most 2 with complex coefficients.

(1) For a complex number m, let T_m be the linear transformation of V given by $T_m(f(x)) = mf(x) - 2f(1)x^2 + f(2)x$. Obtain the representation matrix of T_m with respect to the basis $\{1, x, x^2\}$ of V.

(2) Find all m such that there is a nonzero element $f(x)$ in V satisfying $T_m(f(x)) = 0$, where T_m is the linear transformation defined in (1). Furthermore, for each such m, find all $f(x)$ for which $T_m(f(x)) = 0$.

2 Let P be a plane in $\mathbb{R}^3$ containing the origin and consider the map $\varphi\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ such that each point and its image are symmetric with respect to P.

(1) For $x \in \mathbb{R}^3$, express $\varphi(x)$ using a unit normal vector n of P. Also, show that φ is a linear map.

(2) Find the eigenvalues of φ and eigenspace corresponding to each eigenvalue.

3 Answer the following questions.

(1) Compute the value of the following iterated integral.

$$\int_1^2 dy \int_{\sqrt{y-1}}^1 \frac{xy}{1+x^2} \exp\left(\frac{y^2}{1+x^2}\right) dx + \int_0^1 dx \int_0^1 \frac{xy}{1+x^2} \exp\left(\frac{y^2}{1+x^2}\right) dy$$

(2) For the function f on $\mathbb{R}^2$ given by

$$f(x,y) = (2 + x^2 + y^2)^{xy},$$

determine the polynomial $p(x,y)$ of degree 2 such that the following equation holds.

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - p(x,y)}{x^2 + y^2} = 0$$

4 Answer the following questions.

(1) Suppose that $f(x, y)$ on $\mathbb{R}^2$ is of class C^2 and define a function $F(t)$ on $\mathbb{R}$ by

$$F(t) = f(2 - e^{-t}\cos t, 1 + e^{-t}\sin t).$$

Express $F''(0)$ using the values of the partial derivatives of f.

(2) Find all extremal values of the function $g(x, y) = x^4 + y^4 - 2x^2y^2 + 4x^3 - 4y^3 + 4x^2$ on $\mathbb{R}^2$.