1 Let λ and μ be different complex numbers, and assume that the complex 3×3 square matrix A satisfies

Condition 1: $(A - \lambda I)^2 \neq O, \quad A - \mu I \neq O$

Condition 2: $(A - \lambda I)^2 (A - \mu I) = O.$

Here, I is the identity matrix and O the zero matrix.

(1) Show that λ and μ are eigenvalues of A, and that they are the only eigenvalues of A.

(2) Show that $\text{Ker}\ (A - \lambda I)^2 \cap \text{Ker}\ (A - \mu I) = \{0\}$.

(3) Show that

$$u - \frac{1}{(\mu - \lambda)^2}(A - \lambda I)^2 u \in \text{Ker}\ (A - \lambda I)^2$$

for any $u \in \mathbb{C}^3$.

(4) Show that there is a direct sum decomposition

$$\mathbb{C}^3 = \text{Ker}\ (A - \lambda I)^2 \oplus \text{Ker}\ (A - \mu I).$$

2 Let $f(x,y)$ be a real valued C^2 function on $\mathbb{R}^2$, and for two different points $(a_1,b_1),(a_2,b_2) \in \mathbb{R}^2$ consider the line segment

$$\gamma(t) = \big((1-t)a_1 + ta_2, (1-t)b_1 + tb_2\big) \qquad (0 \leq t \leq 1).$$

Consider the composition $F(t) = f(\gamma(t))$.

(1) Express the second derivative $F''(t)$ in terms of a_1, a_2, b_1, b_2 and the second partial derivatives f_{xx}, f_{xy}, f_{yy} of f.

(2) Show that if f satisfies $f_{xx} > 0$ as well as $f_{xx}f_{yy} - f_{xy}^2 > 0$ in a neighborhood of the line segment $\{\gamma(t)|\ 0 \leq t \leq 1\}$, then $F''(t) > 0$ for any $t \in (0,1)$.

(3) Under the same assumptions as in (2), show that

$$f\left(\frac{a_1+a_2}{2}, \frac{b_1+b_2}{2}\right) < \frac{f(a_1,b_1) + f(a_2,b_2)}{2}.$$

3 Let $R > 0$, and let Γ_R be the path of integration in the complex plane given by running counter-clockwise through the closed curve which consists of

$$\Gamma_{1,R} = [0, R], \qquad \Gamma_{2,R} = \left\{ Re^{i\theta} \middle| \, 0 \le \theta \le \frac{\pi}{4} \right\}, \qquad \Gamma_{3,R} = \{ re^{\pi i/4} | \, 0 \le r \le R \}.$$

(1) Calculate the value of the integral $\displaystyle\int_{\Gamma_R} e^{-z^2} dz$.

(2) Show that $\displaystyle\lim_{R \to \infty} \int_{\Gamma_{2,R}} e^{-z^2} dz = 0$.

(3) Show that the integral $\displaystyle\int_0^\infty e^{-ix^2} dx$ converges, and calculate its value. You can use the identity $\displaystyle\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$.

4 Given non-empty subsets E, F of $\mathbb{R}^2$, let

$$\rho(E,F) = \inf\{\|u-v\| \mid u \in E, v \in F\},$$

where $\|w\| = \sqrt{x^2+y^2}$ for $w=(x,y)$ in $\mathbb{R}^2$.

(1) Show that there are sequences of points $\{u_n\} \subset E$ and $\{v_n\} \subset F$ such that $\lim_{n\to\infty} \|u_n - v_n\| = \rho(E,F)$.

(2) Show that if E is bounded, then there is a strictly increasing sequence of positive integers $\{n_k\}$ such that the subsequences $\{u_{n_k}\}$ and $\{v_{n_k}\}$ of the sequences $\{u_n\}$ and $\{v_n\}$ of part (1) both converge for $k \to \infty$.

(3) Show that if E is bounded as well as closed and F is closed, then

$$\rho(E,F) = \min\{\|u-v\| \mid u \in E, v \in F\}.$$

(4) Give an example (together with the reason) showing that $\min\{\|u-v\| \mid u \in E, v \in F\}$ does not exist even if E, F are both closed.