1 Let a, b, c, d be real numbers and

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a & b & c & d \end{pmatrix}$$

(1) Find the characteristic polynomial of A.

(2) Show that for every eigenvalue of A, the corresponding eigenspace is 1-dimensional.

(3) Find the Jordan normal form for $a = b = -4$, $c = 3$, and $d = 2$. (It is not necessary to find the regular matrix tranforming A into its Jordan normal form).

2

(1) Show that the inequality

$$\log{(n+1)} < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \leq 1 + \log n$$

holds for all positive integers n.

(2) Show that the sequence $\{a_n\}$ with $a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n$ converges for $n \to \infty$.

(3) Show that there are no polynomials with real coefficients $P(X)$, $Q(X)$ such that

$$\frac{P(n)}{Q(n)} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

for every positive integer n.

3 Fix a real number $a > 0$.

(1) For $N_1, N_2 > 0$, and $M > a$, let C be the path of integration given by running counterclockwise through the rectangle with vertices $-N_1$, N_2, $N_2 + iM$, $-N_1 + iM$ in the complex plane. Given a real parameter ξ, find the value of the integral

$$\int_C \frac{e^{i\xi z}}{z - ia}\,dz.$$

(2) For $\xi > 0$, show that the integral

$$\int_{-\infty}^{\infty} \frac{e^{i\xi x}}{x - ia}\,dx$$

converges, and find its value.

(3) For $\xi < 0$, replace the path of integration C of (1) by another suitable one to show that the integral

$$\int_{-\infty}^{\infty} \frac{e^{i\xi x}}{x - ia}\,dx$$

converges, and calculate its value.

4 Given a point $a \in \mathbb{R}^n$ and a real valued function f defined in a neighborhood of a, we say that f is continuous in a if for all $\varepsilon > 0$ there exisits a $\delta = \delta(a) > 0$ such that

$$x \in \mathbb{R}^n,\ \|x - a\| < \delta \Longrightarrow |f(x) - f(a)| < \varepsilon. \tag{A}$$

Here $\|\cdot\|$ is the Euclidean norm, i.e. $\|x\| = \sqrt{\sum_{k=1}^n x_k^2}$ for $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$.

(1) Assuming (A), show that for every $y \in \mathbb{R}^n$ satisfying $\|y - a\| < \dfrac{\delta}{2}$, one has

$$x \in \mathbb{R}^n,\ \|x - y\| < \frac{\delta}{2} \Longrightarrow |f(x) - f(y)| < 2\varepsilon$$

(2) Let f be a real valued function defined in a neighborhood of the closed, bounded set $K \subset \mathbb{R}^n$, and assume that f is continuous in every point $a \in K$. Show that f is uniformly continuous on K, i.e. for all $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$x, y \in K,\ \|x - y\| < \delta \Longrightarrow |f(x) - f(y)| < \varepsilon$$