Consider the following three functions \( u_1(t) = e^t \), \( u_2(t) = te^t \), \( u_3(t) = \frac{t^2}{2} e^t \) defined on \( \mathbb{R} \).

(1) Let \( V \) be the real vector space \( V \) of real valued \( C^\infty \)-functions on \( \mathbb{R} \). Show that 
\[ \{ u_1(t), u_2(t), u_3(t) \} \]
are linear independent as elements \( V \).

(2) Let \( W \) be the \( \mathbb{R} \)-subvector space of \( V \) generated by \( u_1(t), u_2(t), u_3(t) \). Verify that \( \frac{d}{dt} \) is a linear map from \( W \) to \( W \), and calculate the representing matrix \( A \) with respect to the basis \( \{ u_1(t), u_2(t), u_3(t) \} \).

(3) Prove that the solution space of the differential equation
\[
\frac{d^3 u}{dt^3} - 3 \frac{d^2 u}{dt^2} + 3 \frac{du}{dt} - u = 0
\]
contains the 3-dimensional vector space spanned by \( u_1(t), u_2(t), u_3(t) \).

(4) Prove that if \( u(t) = C(t)e^t \) is a solution of the differential equation
\[
\frac{d^3 u}{dt^3} - 3 \frac{d^2 u}{dt^2} + 3 \frac{du}{dt} - u = 0,
\]
then \( C(t) \) is a polynomial of degree at most 2.

(5) Determine the space of solutions of the differential equation
\[
\frac{d^3 u}{dt^3} - 3 \frac{d^2 u}{dt^2} + 3 \frac{du}{dt} - u = 0.
\]
Define the functions \( \phi_n (n = 1, 2, \ldots) \) on \([0, \infty)\) by \( \phi_n(x) = n^2 xe^{-nx} \).

(1) Calculate \( \int_0^\infty \phi_n(x) \, dx \).

(2) Show that, for any \( \delta > 0 \), the functions \( \{\phi_n\} \) converge uniformly to 0 on \([\delta, \infty)\).

(3) Show that for any bounded, continuous function \( f \) on \([0, \infty)\),

\[
\lim_{n \to \infty} \int_0^\infty f(x)\phi_n(x) \, dx = f(0)
\]

holds.
3. Answer the following questions

1. Assume that the function $f(z)$ is regular on a domain containing the disk $D_R = \{ |z \in \mathbb{C}, |z| \leq R \}$. Prove that if $z \in \mathbb{C}$ lies in the disc $D_R$, then

$$f'(z) = \frac{1}{2\pi i} \int_{|\zeta|=R} \frac{f(\zeta)}{(\zeta - z)^2} d\zeta.$$

2. Use (1) to prove that a regular function $f(z)$, which is bounded on the whole complex plane, satisfies $f'(z) \equiv 0$.

3. Determine the subset of the $z$-plane which maps under the regular function $w = e^z$ to the domain $\{ w \in \mathbb{C}, |w| < a \} \ (a > 0)$ of the $w$-plane, and graph it.

4. Show that a regular function defined on the whole complex plane whose real part is non-positive is a constant function.
For a subset $M$ of $\mathbb{R}^n$ and a point $x$ of $\mathbb{R}^n$ define

$$d(x, M) = \inf\{|x - y| : y \in M\}.$$ 

Here $|x|$ is the Euclidean norm, i.e. for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ we define $|x| = \sqrt{\sum_{i=1}^{n} x_i^2}$.

1. Show that $d(x, M) = 0$ is a necessary and sufficient condition for $x \in M$.

2. Show that $d(x, M) \leq |y - z| + |x - y|$ for any two points $x, y$ in $\mathbb{R}^n$, and any point $z$ in $M$.

3. Show that for fixed $M$, the function $x \mapsto d(x, M)$ is continuous on $\mathbb{R}^n$.

4. Show that if $M$ is closed, then for any $x \in \mathbb{R}^n$ there is a $y^* \in M$ such that

$$|x - y^*| = d(x, M).$$