1. Let $a$ be a real number, and $A = A_a$ be the matrix

$$A_a = \begin{pmatrix}
a & 2 - 2a & -2 + 2a \\
-1 - a & -3 + 2a & 2 - 2a \\
-1 - a & -3 + a & 2 - a
\end{pmatrix}.$$ 

(1) Show that $A$ is diagonalizable if $a = 1$.

(2) Determine the Jordan normal form of $A$ for $a \neq 1$.

(3) Let $\langle \cdot, \cdot \rangle$ be the canonical Euclidean inner product. Determine for which $a$ the sequence $\{ \langle x, A^n x \rangle \}_{n=1}^{\infty}$ is bounded for any $x, y \in \mathbb{R}^3$. 

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2 Let $f(x, y)$ be a $C^2$-function, and $p(x, y)$ be a polynomial of degree at most 2. Assume that
\[
\lim_{(x, y) \to (0, 0)} \frac{f(x, y) - p(x, y)}{x^2 + y^2} = 0
\]

(1) Express $p(x, y)$ in terms of $f$ and its (higher) partial derivatives (the answer suffices).

(2) Calculate $\frac{1}{2\pi} \int_0^{2\pi} p(r \cos \theta, r \sin \theta) d\theta$ for fixed $r > 0$.

(3) Show that if $f(0, 0) = \frac{1}{2\pi} \int_0^{2\pi} f(r \cos \theta, r \sin \theta) d\theta$ holds for every $r > 0$, then $f_{xx}(0, 0) + f_{yy}(0, 0) = 0$. 
3) Given a natural number $n \geq 2$ and a real number $R > 0$, let $\gamma_{1,R} = [0, R]$, $\gamma_{2,R} = \{Re^{i\theta} | 0 \leq \theta \leq 2\pi/n\}$, $\gamma_{3,R} = \{r^{2\pi i / n} | 0 \leq r \leq R\}$. Let $\gamma_R$ be the closed oriented curve given by running through $\gamma_{1,R}$, $\gamma_{1,R}$, and $\gamma_{1,R}$ counterclockwise.

(1) For $R > 1$, calculate the integral $\int_{\gamma_R} \frac{dz}{z^n + 1}$.

(2) Show that $\lim_{R \to \infty} \int_{\gamma_{2,R}} \frac{dz}{z^n + 1} = 0$.

(3) Calculate the integral $I_n = \int_0^\infty \frac{dx}{x^n + 1}$.

(4) Calculate $\lim_{n \to \infty} I_n$. 
Given real numbers $a_0, a_1, a_2$ and a positive real number $a_3$ let $f(x, y) = f_{a_0, a_1, a_2, a_3}(x, y)$ be the polynomial

$$f(x, y) = a_0 x^3 + a_1 x^2 y + a_2 xy^2 + a_3 y^3.$$ 

Let $C = C(a_0, a_1, a_2, a_3)$ be the subset of $\mathbb{R}^2$ given by

$$C = \{(x, y) \mid x, y \geq 0, f(x, y) = 1\}$$

(1) If $C$ is bounded, then the function $x^2 + y^2$ achieves its maximum on $C$. Explain why.

(2) Let $\alpha \geq 0$. Determine, for which $a_0, a_1, a_2, a_3, \alpha$, the set $C$ intersects the line $y = \alpha x$. Determine the intersection point(s) if any exists.

(3) Let $\mathcal{L}$ be the set of all lines $y = \alpha x$ with positive slope $\alpha \geq 0$. Show that $C$ intersects every line $L \in \mathcal{L}$ if and only if $C$ is bounded.

(4) Let $A$ be the subset $\mathbb{R}^4$ consisting of those $(a_0, a_1, a_2, a_3)$ such that $C$ is bounded. Show that $A$ is an open subset of $\mathbb{R}^4$. 

(28.7.2012) the end