Let $V \subset \mathbb{R}^4$ be the subspace spanned by the vectors
\[
\begin{pmatrix} 2 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix},
\]
and let $W_t \subset \mathbb{R}^4$ be the subspace spanned by the vectors
\[
\begin{pmatrix} t + 1 \\ t - 1 \\ t + 1 \\ t - 1 \end{pmatrix}, \quad \begin{pmatrix} t + 2 \\ t \\ t + 1 \\ t + 1 \end{pmatrix}, \quad \begin{pmatrix} t + 3 \\ t + 1 \\ t + 2 \\ t + 1 \end{pmatrix},
\]
Here $t$ is a real number. Please answer the following questions.

(1) Solve the system of linear equations
\[
\begin{pmatrix} a & b & c & d \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]
with respect to the real numbers $a, b, c, d$.

(2) Find a system of linear equations whose space of solutions is equal to $V$.

(3) Find a system of linear equations whose space of solutions is equal to $W_t$.

(4) Find a basis for the subspace $V \cap W_t$ and find its dimension.
Let $T : \mathbb{C}^3 \rightarrow \mathbb{C}^3$ be the complex linear map defined by $T(v) = Av$, where

$$A = \begin{pmatrix} -5 & 6 & 2 \\ -8 & 8 & 1 \\ 8 & -6 & 1 \end{pmatrix}.$$ 

Please answer the following questions.

(1) Find the eigenvalues and eigenvectors for $T$.

(2) Let $V_\lambda \subset \mathbb{C}^3$ be the eigenspace corresponding to the eigenvalue $\lambda$ for $T$. Show that $\mathbb{C}^3$ decomposes as the direct sum of the subspaces $V_\lambda$.

(3) Let $v_\lambda \in V_\lambda$ be the summand of $v \in \mathbb{C}^3$ corresponding to the eigenvalue $\lambda$, and let $p_\lambda : \mathbb{C}^3 \rightarrow \mathbb{C}^3$ be the linear map defined by $p_\lambda(v) = v_\lambda$. For each eigenvalue $\lambda$, find the matrix that represents $p_\lambda$ with respect to the standard basis of $\mathbb{C}^3$. 

(July 23rd, 2011)
Please answer the following questions.

(1) Find the 4th order Taylor expansion at the origin and with respect to \( x, y \) of the two-variable function \( \cos \left( \frac{x}{1 + y^2} \right) \).

(2) Decide whether or not the function \( f(x, y) = x^2 + 2x^2y - xy^2 \ (x, y \in \mathbb{R}) \) has any extrema.

(3) Find the value of the integral

\[
\int \int_{D} \frac{1}{1 + (x^2 + y^2)^2} \, dx \, dy,
\]

where \( D \) is the planar region defined by

\[
D = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq y\}.
\]
Let $a, b, n$ be positive integers with $a < b$ and define

$$S_n(a, b) = \frac{1}{an + 1} + \frac{1}{an + 2} + \cdots + \frac{1}{bn - 1} + \frac{1}{bn}.$$ 

Please answer the following questions.

1. Show that $\log \frac{bn + 1}{an + 1} < S_n(a, b) < \frac{b}{a}$.

2. Find the value of $S(a, b) = \lim_{n \to \infty} S_n(a, b)$.

3. With $S(a, b)$ as in (2), decide whether the series

$$\sum_{k=1}^{\infty} S(k^2, k^2 + 1)$$

converges or diverges.