Entrance Examination for the Ph. D. Program
Graduate School of Mathematics
Nagoya University
2012 Admission

Part 2 of 2

Tuesday, February 7, 2012, 13:00 p.m.∼16:00 p.m.

Note:

1. Please do not turn pages until told to do so.
2. The problem sheet consists of the cover page and 4 single-sided pages. After the exam has begun, please first confirm that the number of pages and their printing and order are correct. Please report any problem immediately.
3. There are a total of 4 problems labeled [1], [2], [3], and [4], respectively. Please **answer all 4 problems.**
4. The answering sheet consists of 4 single-sided pages. Please **confirm the number of pages**, and please **do not remove the staple**.
5. Please write the answers to problems [1], [2], [3], and [4] on pages [1], [2], [3], and [4] of the answering sheet, respectively.
6. **Please write name and application number in the space provided on each of the 4 pages in the answering sheet.**
7. The back side of the 4 pages in the answering sheet may also be used. If used, please check the box at the lower right-hand corner on the front side.
8. If the answering sheet staple is torn, or if additional paper is needed for calculations, please notify the exam proctor.
9. After the exam has ended, please hand in the 4 page answering sheet. The problem sheet and any additional sheets used for calculations may be taken home.

Notation:

The symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of integers, rational numbers, real numbers, and complex numbers, respectively.

1 For integers $n \geq 2$, let $M_n(\mathbb{C})$ denote the set of all complex $n \times n$-matrices. Please answer the following questions.

(1) Let $D \in M_n(\mathbb{C})$ be a fixed diagonal matrix whose diagonal entries are pairwise distinct. Show that if $X \in M_n(\mathbb{C})$ satisfies $DX = XD$, then X is a diagonal matrix.

(2) Let $A \in M_n(\mathbb{C})$ have n pairwise distinct eigenvalues and let $m \geq 2$ be an integer. Find the number of solutions $X \in M_n(\mathbb{C})$ to the equation $X^m = A$.

2 For a fixed positive real number a, let $f(x,y)$ be the function on $\mathbb{R}^2$ defined by

$$f(x,y) = \begin{cases} (x^2+y^2)^a \sin \frac{1}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Please answer the following questions.

(1) Show that $f(x,y)$ is continuous at $(0,0)$.

(2) Find all a for which the partial derivatives $f_x(0,0)$ and $f_y(0,0)$ exist. For all such a, find the value of the partial derivative $f_x(0,0)$.

(3) Find all a for which the function $f(x,y)$ is of class C^1 on $\mathbb{R}^2$.

3 Let $D = \{z \in \mathbb{C} \mid |z| < 1\}$ be the open unit disc in the complex plane, and let $f(z)$ be the following power series of the complex variable z:

$$f(z) = z + z^2 + z^4 + z^8 + \cdots + z^{2^n} + \cdots .$$

Please answer the following questions.

(1) Show that the radius of convergence of $f(z)$ is equal to 1.

(2) By considering the limit $\lim_{x \to 1^-} f(x)$ as $0 < x < 1$ approaches 1 from the left, show that there does not exist a holomorphic function $g(z)$ defined on a neighborhood U of $z = 1$ such that $f(z) = g(z)$ for all $z \in U \cap D$.

(3) Let a be a complex number whose absolute value is equal to 1. Show that there does not exist a holomorphic function $g(z)$ defined on a neighborhood V of $z = a$ such that $f(z) = g(z)$ for all $z \in V \cap D$.

(Hint: If k, m are positive integers and $z = e^{2\pi i k/2^m}$, then $z^{2^n} = 1$ for all $n \geq m$.)

4 Let X be a set and let $f\colon X \to X$ be a map. If C is a subset of X, then $f(C)$ denotes the image of C by f. Decide if each of the following statements is true or false. If true, then please give a proof. If false, then please give a counterexample.

(1) If $f \circ f = f$, then f is the identity map. Here, $f \circ f$ denotes the composition of the map f with itself.

(2) If there exists a map $g\colon X \to X$ such that $g \circ f$ is the identity map, then also $f \circ g$ is the identity map. Here, $f \circ g$ and $g \circ f$ denote the composite maps.

(3) If A and B are subsets of X with the property that $f(A) \subset A$ and $f(B) \subset B$, then $f(A \cap B) \subset A \cap B$.

(4) If A and B are subsets of X with the property that $f(A) \supset A$ and $f(B) \supset B$, then $f(A \cap B) \supset A \cap B$.