1. Let $V \subset \mathbb{R}^4$ be the subspace generated by the vectors

\[
\begin{pmatrix}
1 \\
-1 \\
1 \\
-1
\end{pmatrix},
\begin{pmatrix}
1 \\
-1 \\
1 \\
-1
\end{pmatrix},
\begin{pmatrix}
t + 4 \\
t \\
t + 2 \\
t
\end{pmatrix},
\begin{pmatrix}
t + 3 \\
t \\
t + 1 \\
t + 1
\end{pmatrix},
\begin{pmatrix}
t + 2 \\
t \\
t + 2 \\
t + 2
\end{pmatrix}
\]

and for every real number $t$, let $W \subset \mathbb{R}^4$ be the subspace generated by the three vectors

\[
\begin{pmatrix}
t + 4 \\
t \\
t + 2 \\
t
\end{pmatrix},
\begin{pmatrix}
t + 3 \\
t \\
t + 1 \\
t + 1
\end{pmatrix},
\begin{pmatrix}
t + 2 \\
t \\
t + 2 \\
t + 2
\end{pmatrix}
\]

Please answer the following problems.

(1) Find the dimension of $W$.

(2) Find the dimension of $V + W$.

(3) Find the dimension of $V \cap W$. 
Consider the set

\[ V = \{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{R}, \ a + d = 0 \} \]

as a real vector space with respect to matrix sum and scalar product. Please answer the following problems.

1. Show that \( \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\} \) is a basis of \( V \).

2. Let \( A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \) be an element of \( V \). Show that the formula

\[ F_A(X) = AX -XA \]

defines a linear map \( F_A : V \to V \). In addition, find the matrix that represents \( F_A \) with respect to the basis in (1).

3. For \( A = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix} \), find the eigenvalues and eigenvectors of \( F_A \).
3 Please answer the following problems.

(1) Find the value of the double integral

\[ \int_D xy \, dx \, dy, \]

where \( D = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, \ y \geq 0, \ \sqrt{x} + \sqrt{y} \leq 1\} \).

(2) Let \( z = f(x, y) \) be the real valued function of class \( C^1 \) defined implicitly by the equation

\[ x^2 y + yz + z^3 x = 3 \]

on the open disc of radius \( \frac{1}{2} \) centered at \((1, 1)\). Find the partial derivatives

\[ \frac{\partial f}{\partial x}(1, 1), \quad \frac{\partial f}{\partial y}(1, 1) \]

of \( f(x, y) \) at \((x, y) = (1, 1)\).

(3) Find the Taylor expansion of the function \( g(x) = \frac{1}{\cos x} \) to the 4th order around \( x = 0 \).
Let \( a \) be a real number and let \( f: \mathbb{R}^2 \rightarrow \mathbb{R} \) be the function defined by
\[
f(x, y) = xy + ay^2 - x^3.
\]
Find the maximum value of \( f(x, y) \) on \( \mathbb{R}^2 \) and find the point(s) where this maximum value is attained.