1 Let V be a $\mathbb{C}$-vector space of finite dimension n, and let $f : V \to V$ be a linear map for which there exists an integer $k \geq 1$ such that $f^k = 0$. Please answer the following questions.

(1) Show that 0 is the only eigenvalue of f.

(2) For $n = 3$, what is the Jordan normal form of f?

(3) Show that $I_V + f$ is a bijection. Here I_V denotes the identity map of V.

2 Let $u(x, y)$ be a real function of class C^2 on $\mathbb{R}^2$ that satisfies

$$u(x+m, y+n) = u(x, y) \quad (\ \forall (m, n) \in \mathbb{Z}^2 \ ),$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Please answer the following questions.

(1) Show that $u(x, y)$ is a bounded function on $\mathbb{R}^2$.

(2) Let $v(x, y)$ be the real function on $\mathbb{R}^2$ defined by

$$v(x, y) = -\int_0^x \frac{\partial u}{\partial y}(s, y)ds + \int_0^y \frac{\partial u}{\partial x}(0, t)dt.$$

Show that $v(x, y)$ is of class C^1 and satisfies

$$(-\frac{\partial u}{\partial y}(x, y), \frac{\partial u}{\partial x}(x, y)) = (\frac{\partial v}{\partial x}(x, y), \frac{\partial v}{\partial y}(x, y)).$$

(3) Let z be a complex number with real part x and imaginary part y. Let $v(x, y)$ be the function from (2) and define

$$f(z) = u(x, y) + iv(x, y).$$

Show that $f(z)$ is a holomorphic function on all of $\mathbb{C}$.

(4) Show that $f(z)$ is a constant function and conclude that also $u(x, y)$ is a constant function.

3 In the complex plane, let C_ρ the upper semi-circle of radius $\rho > 0$,

$$C_\rho = \left\{\rho e^{i\theta} \mid 0 \le \theta \le \pi\right\}$$

and let C_ρ be given the counter-clockwise orientation. Let $f(z)$ be the meromorphic function on $\mathbb{C}$ defined by

$$f(z) = \frac{e^{3iz} - 3e^{iz}}{z^3}.$$

Please answer the following questions.

(1) Fix $r > 0$. For $R > r$, let γ_R be the oriented closed curve comprised of the following four curve segments oriented as indicated.

$\gamma_{1,R} = [r, R]$ (from r to R),

$\gamma_{2,R} = C_R$ (from R to $-R$),

$\gamma_{3,R} = [-R, -r]$ (from $-R$ to $-r$),

$\gamma_{4,R} = C_r$ (with the opposite orientation from $-r$ to r)

Evaluate the complex line integral

$$\int_{\gamma_R} f(z)dz.$$

(2) For $r > 0$, show the equality

$$\int_r^\infty \frac{\sin^3 x}{x^3} dx = \frac{i}{8} \int_{C_r} f(z)dz,$$

where the integral on the left-hand side is the improper integral.

(3) Evaluate the improper integral

$$\int_0^\infty \frac{\sin^3 x}{x^3} dx.$$

4 Let X and Y be sets, and let $F : X \to Y$ be a map. Please answer the following questions.

(1) For subsets $A \subset X$ and $B \subset Y$, let

$$F(A) = \{F(x) \mid x \in A\},\ F^{-1}(B) = \{x \in X \mid F(x) \in B\}.$$

Decide if the following assertions (a)-(b) are true or false. If true, give a proof; if false, give a counter-example.

(a) $F(A_1 \cap A_2) = F(A_1) \cap F(A_2)$.

(b) $F^{-1}(B_1 \cap B_2) = F^{-1}(B_1) \cap F^{-1}(B_2)$.

(2) If X is not the empty set, and if $F : X \to Y$ is injective, show that there exists a surjective map from Y to X.