Let $V$ and $W$ be finite dimension linear spaces over $\mathbb{C}$, and $f : V \to W$ a linear map. Answer the following questions:

1. Show that, when $f$ is surjective, there exists a linear map $g : W \to V$ such that the composition $fg$ of $g$ and $f$ is the identity on $W$.

2. Show that, when $f$ is injective, there exists a linear map $g : W \to V$ such that $gf$ is the identity on $V$.

3. Show that, for any $f$, there exists a linear map $g : W \to V$ such that $f = fgf$ and $g = gfg$.

4. Assume that $g$ satisfies the equations of (3). Show that $f$ is a bijection from $\text{Im } g$ (the image of $g$) to $\text{Im } f$ (the image of $f$).
We pose $U = \mathbb{R}^2 \setminus \{(0,0)\}$, the plane minus the origin. The functions on $U$ will be real-valued. We say that a function $h(x,y)$ on $U$ is harmonic, if it is of class $C^2$, and satisfies $\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0$ everywhere. Answer the following questions:

(1) For any continuous function $f(x,y)$ on $U$, prove the following equations:

$$\int_{-r}^{r} \left\{ f(\sqrt{r^2-y^2}, y) - f(-\sqrt{r^2-y^2}, y) \right\} dy = r \int_0^{2\pi} f(r \cos \theta, r \sin \theta) \cos \theta \, d\theta,$$

$$\int_{-r}^{r} \left\{ f(x, \sqrt{r^2-x^2}) - f(x, -\sqrt{r^2-x^2}) \right\} dx = r \int_0^{2\pi} f(r \cos \theta, r \sin \theta) \sin \theta \, d\theta.$$

Here we assume $r > 0$.

(2) For any function $g(x,y)$ of class $C^1$ on $U$, prove the following equations:

$$\iint_{e^2 \leq x^2 + y^2 \leq r^2} \frac{\partial g}{\partial x}(x,y) \, dxdy = r \int_0^{2\pi} g(r \cos \theta, r \sin \theta) \cos \theta \, d\theta,$$

$$- \varepsilon \int_0^{2\pi} g(\varepsilon \cos \theta, \varepsilon \sin \theta) \cos \theta \, d\theta,$$

$$\iint_{e^2 \leq x^2 + y^2 \leq r^2} \frac{\partial g}{\partial y}(x,y) \, dxdy = r \int_0^{2\pi} g(r \cos \theta, r \sin \theta) \sin \theta \, d\theta,$$

$$- \varepsilon \int_0^{2\pi} g(\varepsilon \cos \theta, \varepsilon \sin \theta) \sin \theta \, d\theta.$$

Here we assume $0 < \varepsilon < r$.

(3) For any harmonic function $h(x,y)$ on $U$, let $H(\rho, \theta) = h(\rho \cos \theta, \rho \sin \theta)$ and

$$F(\rho) = \rho \int_0^{2\pi} \frac{\partial H}{\partial \rho}(\rho, \theta) \, d\theta,$$

for $\rho > 0$. Show that $F(\rho)$ is constant.

(4) Find all the harmonic functions on $U$ whose value only depends on the distance from the origin.
Let $C_r$ be the upper half of the circle of radius $r > 0$ in the complex plane:

$$C_r = \{ re^{i\theta} \mid 0 \leq \theta \leq \pi \}.$$

$C_r$ is oriented counter-clockwise. Answer the following questions:

(1) Evaluate the limit

$$\lim_{R \to \infty} \int_{C_R} \frac{1 + iz - e^{iz}}{z^3} \, dz.$$

(2) Show that any regular function $g(z)$ on $\mathbb{C}$ satisfies the equation

$$\lim_{\varepsilon \to 0} \int_{C_{\varepsilon}} g(z) \, dz = 0.$$

Here we use the right limit, obtained by restricting $\varepsilon$ to positive values.

(3) Using the right limit as in (2), evaluate

$$\lim_{\varepsilon \to 0} \int_{C_{\varepsilon}} \frac{1 + iz - e^{iz}}{z^3} \, dz.$$

(4) Evaluate the generalized integral

$$\int_{0}^{\infty} \frac{x - \sin x}{x^3} \, dx.$$
4 Answer the following questions:

(1) For any topological space $X$, show that the following properties (a) and (b) are equivalent:

(a) $X$ has a smallest non-empty closed set.

(b) $X$ is not empty, and for any $\{U_\lambda\}_{\lambda \in \Lambda}$ open cover of $X$, there is a $\lambda \in \Lambda$ such that $X = U_\lambda$.

(2) Let $X$ and $Y$ be topological spaces, $f : X \to Y$ a continuous map between them. Show that if $X$ satisfies the condition (a) above, then its image $f(X)$ is again a topological space that satisfies (a).