1 Let V and W be finite dimension linear spaces over $\mathbb{C}$, and $f : V \to W$ a linear map. Answer the following questions:

(1) Show that, when f is surjective, there exists a linear map $g : W \to V$ such that the composition fg of g and f is the identity on W.

(2) Show that, when f is injective, there exists a linear map $g : W \to V$ such that gf is the identity on V.

(3) Show that, for any f, there exists a linear map $g : W \to V$ such that $f = fgf$ and $g = gfg$.

(4) Assume that g satisfies the equations of (3). Show that f is a bijection from $\operatorname{Im} g$ (the image of g) to $\operatorname{Im} f$ (the image of f).

2 We pose $U = \mathbb{R}^2 \setminus \{(0,0)\}$, the plane minus the origin. The functions on U will be real-valued. We say that a function $h(x,y)$ on U is *harmonic*, if it is of class C^2, and satisfies $\dfrac{\partial^2 h}{\partial x^2} + \dfrac{\partial^2 h}{\partial y^2} = 0$ everywhere. Answer the following questions:

(1) For any continuous function $f(x,y)$ on U, prove the following equations:

$$\int_{-r}^{r} \left\{ f(\sqrt{r^2-y^2}, y) - f(-\sqrt{r^2-y^2}, y) \right\} dy = r \int_0^{2\pi} f(r\cos\theta, r\sin\theta)\cos\theta \, d\theta,$$

$$\int_{-r}^{r} \left\{ f(x, \sqrt{r^2-x^2}) - f(x, -\sqrt{r^2-x^2}) \right\} dx = r \int_0^{2\pi} f(r\cos\theta, r\sin\theta)\sin\theta \, d\theta.$$

Here we assume $r > 0$.

(2) For any function $g(x,y)$ of class C^1 on U, prove the following equations:

$$\iint_{\varepsilon^2 \le x^2+y^2 \le r^2} \frac{\partial g}{\partial x}(x,y)\, dxdy = r \int_0^{2\pi} g(r\cos\theta, r\sin\theta)\cos\theta \, d\theta - \varepsilon \int_0^{2\pi} g(\varepsilon\cos\theta, \varepsilon\sin\theta)\cos\theta \, d\theta,$$

$$\iint_{\varepsilon^2 \le x^2+y^2 \le r^2} \frac{\partial g}{\partial y}(x,y)\, dxdy = r \int_0^{2\pi} g(r\cos\theta, r\sin\theta)\sin\theta \, d\theta - \varepsilon \int_0^{2\pi} g(\varepsilon\cos\theta, \varepsilon\sin\theta)\sin\theta \, d\theta.$$

Here we assume $0 < \varepsilon < r$.

(3) For any harmonic function $h(x,y)$ on U, let $H(\rho,\theta) = h(\rho\cos\theta, \rho\sin\theta)$ and $F(\rho) = \rho \displaystyle\int_0^{2\pi} \frac{\partial H}{\partial \rho}(\rho,\theta)\, d\theta$, for $\rho > 0$. Show that $F(\rho)$ is constant.

(4) Find all the harmonic functions on U whose value only depends on the distance from the origin.

3 Let C_r be the upper half of the circle of radius $r > 0$ in the complex plane:

$$C_r = \{re^{i\theta} \mid 0 \le \theta \le \pi\}.$$

C_r is oriented counter-clockwise. Answer the following questions:

(1) Evaluate the limit

$$\lim_{R\to\infty} \int_{C_R} \frac{1 + iz - e^{iz}}{z^3}\, dz.$$

(2) Show that any regular function $g(z)$ on $\mathbb{C}$ satisfies the equation

$$\lim_{\varepsilon\to 0} \int_{C_\varepsilon} g(z)\, dz = 0.$$

Here we use the right limit, obtained by restricting ε to positive values.

(3) Using the right limit as in (2), evaluate

$$\lim_{\varepsilon\to 0} \int_{C_\varepsilon} \frac{1 + iz - e^{iz}}{z^3}\, dz.$$

(4) Evaluate the generalized integral

$$\int_0^\infty \frac{x - \sin x}{x^3}\, dx.$$

4 Answer the following questions:

(1) For any topological space X, show that the following properties (a) and (b) are equivalent:

(a) X has a smallest non-empty closed set.

(b) X is not empty, and for any $\{U_\lambda\}_{\lambda \in \Lambda}$ open cover of X, there is a $\lambda \in \Lambda$ such that $X = U_\lambda$.

(2) Let X and Y be topological spaces, $f : X \to Y$ a continuous map between them. Show that if X satisfies the condition (a) above, then its image $f(X)$ is again a topological space that satisfies (a).