1 With respect to a complex number a, we consider the matrix

$$A = \begin{pmatrix} -4 & 2 & -2 \\ -2-a & 1+a & -1 \\ 2-a & -1+a & 1 \end{pmatrix}.$$

Answer the following questions:

(1) Find all the values of a such that $\operatorname{rank} A = 1$.

(2) When $\operatorname{rank} A = 1$, find a regular matrix P such that $P^{-1}AP$ is diagonal.

(3) Find all the values of a such that A cannot be diagonalized.

2 Let V be the linear space of polynomials of order 3 or less over $\mathbb{R}$:

$$V = \{p + qx + rx^2 + sx^3 \mid p, q, r, s \in \mathbb{R}\}.$$

For any $f(x) \in V$, we define

$$T(f(x)) = \frac{1}{x-1} \int_1^x (t-1) f'(t)\, dt.$$

Answer the following questions:

(1) Show that the mapping T, which maps $f(x) \in V$ to $T(f(x))$, is a linear map from V to V.

(2) Find the representation matrix of T, using $1, x, x^2, x^3$ as a basis for V.

(3) Assuming $g(x) = p + qx + rx^2 + sx^3 \in V$ $(p, q, r, s \in \mathbb{R})$, find a necessary and sufficient condition on p, q, r, s so that there exists an $f(x) \in V$ such that $T(f(x)) = g(x)$.

(4) For some constant $k \in \mathbb{R}$, we fix $g(x) = 1 - x + kx^2 - 3x^3 \in V$. Check whether $T(f(x)) = g(x)$ admits a solution $f(x) \in V$. If there are solutions, find all such $f(x) \in V$.

3 Answer the following questions:

(1) For the 2-variable function $f(x,y) = e^{(x+y)\cos(x-y)}$, find the 2nd order polynomial $p(x,y)$ such that

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - p(x,y)}{x^2 + y^2} = 0.$$

(2) Let $g(x,y)$ be a function of class C^1 around $(x,y) = (a,b)$, we define

$$c = g(a,b), \quad \xi = \frac{\partial g}{\partial x}(a,b), \quad \eta = \frac{\partial g}{\partial y}(a,b).$$

Let $F(u,v,w)$ be a function of class C^1 around $(u,v,w) = (a,b,c)$, we define

$$\alpha = \frac{\partial F}{\partial u}(a,b,c), \quad \beta = \frac{\partial F}{\partial v}(a,b,c), \quad \gamma = \frac{\partial F}{\partial w}(a,b,c).$$

Let $G(x,y) = (x, y, g(x,y))$ and $H(x,y) = (F \circ G)(x,y)$. Represent $\dfrac{\partial H}{\partial x}(a,b)$ and $\dfrac{\partial H}{\partial y}(a,b)$ using $\alpha, \beta, \gamma, \xi, \eta$. Here, $F \circ G$ is the composition of G and F.

(3) Evaluate the following integral:

$$\int_0^2 dy \int_{\sqrt{y}}^{\sqrt{2}} \exp\left(\frac{y}{x}\right) dx.$$

4 Answer the following questions:

(1) For $\alpha > 0$ and $n = 1, 2, \ldots$, show that

$$\frac{1}{(n+1)^\alpha} \leq \int_n^{n+1} \frac{dx}{x^\alpha}.$$

(2) Show that, when $\alpha > 1$, the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+1)^\alpha}$ converges.

(3) Show that, when $\alpha \to \infty$,

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)^\alpha} \to 0.$$

(4) We assume that $\alpha > 1$, the sequence $\{a_n\}_{n=1}^{\infty}$ is such that $a_n > 0$ $(n = 1, 2, \ldots)$, and the series $\displaystyle\sum_{n=1}^{\infty} a_n$ converges. Under those assumptions, show that both $\displaystyle\sum_{n=1}^{\infty} a_n^{\ n}$ and $\displaystyle\sum_{n=1}^{\infty} \left(a_n + \frac{1}{(n+1)^\alpha}\right)^n$ do converge.

(5) Under the assumptions of (4), show that, when $\alpha \to \infty$,

$$\sum_{n=1}^{\infty} \left\{ \left(a_n + \frac{1}{(n+1)^\alpha}\right)^n - a_n^{\ n} \right\} \longrightarrow 0.$$