

# Supersingular Abelian Varieties and Related Arithmetic

September 30th – October 4th, 2019

1F Seminar Room, Science South Bldg., Nagoya University

## PROGRAM

### September 30th (Mon)

- 09:50–10:00 — *Opening* —
- 10:00–11:00 **Tomoyoshi Ibukiyama** (*Osaka University*)  
A survey on quaternion hermitian lattices and algebraic modular forms
- 11:20–12:20 **Chia-Fu Yu** (*Academia Sinica*)  
A survey on Dieudonné modules and an arithmetic application
- *Lunch Break* —
- 13:50–14:50 **Toshiyuki Katsura** (*University of Tokyo*)  
On the supersingular locus of the moduli space of principally polarized abelian varieties in positive characteristic
- 15:10–16:10 **Tomoyoshi Ibukiyama** (*Osaka University*)  
A survey on polarizations of superspecial abelian varieties and the field of definition
- 16:30–17:30 **Jiangwei Xue** (*Wuhan University*)  
A survey on quaternion orders and selectivity
- 18:00– — *Reception at Restaurant “Hananoki”* —

### October 1st (Tue)

- 10:00–11:00 **Valentijn Karemaker** (*Utrecht University & Stockholm University*)  
Arithmetic of twists of supersingular abelian varieties
- 11:30–12:30 **Jiangwei Xue** (*Wuhan University*)  
On counting certain principally polarized superspecial abelian surfaces over  $\mathbb{F}_p$
- *Lunch Break* —
- 14:00–15:00 **Rachel Pries** (*Colorado State University*)  
Curves of high genus with many slopes of  $1/2$
- 15:30–16:30 **Yasuhiro Oki** (*University of Tokyo*)  
On certain moduli spaces of polarized supersingular QM abelian 4-folds

## October 2nd (Wed)

10:00–11:00 **Pol van Hoften** (*King's College London*)  
A geometric Jacquet–Langlands correspondence for paramodular Siegel threefolds

11:30–12:30 **Jeff Achter** (*Colorado State University*)  
Arithmetic occult periods

Afternoon — *Private Discussion* —

## October 3rd (Thu)

10:00–11:00 **Christophe Ritzenthaler** (*Université de Rennes 1*)  
Reduction of curves and abelian varieties

11:30–12:30 **Ken-ichi Sugiyama** (*Rikkyo University*)  
On a generalization of Deuring's results

— *Lunch Break* —

14:00–15:00 **Momonari Kudo** (*Kobe City College of Technology & Kyushu University*)  
Computational approaches to superspecial curves of genera 4 and 5 over finite fields

15:30–16:30 **Shushi Harashita** (*Yokohama National University*)  
The existence of supersingular curves of genus 4 in arbitrary characteristic

## October 4th (Fri)

10:00–11:00 **Tomoyoshi Ibukiyama** (*Osaka University*)  
Supersingular abelian surfaces, automorphisms, and the trace formulas

11:30–12:30 **Chia-Fu Yu** (*Academia Sinica*)  
Mass formula for principally polarized supersingular abelian 3-folds

# ABSTRACT

**Tomoyoshi Ibukiyama** (*Osaka University*) ..... September 30th (Mon), 10:00–11:00

## A survey on quaternion hermitian lattices and algebraic modular forms

We give an elementary and explicit introduction on fundamental facts on quaternion algebras  $D$  over  $\mathbb{Q}$  and positive definite quaternion hermitian lattices in  $D^n$ . In particular, we define classes and genera of maximal quaternion hermitian lattices and description of corresponding quaternion hermitian matrices. We also define associated automorphic forms (called algebraic modular forms) and the theory of Hecke operators.

**Chia-Fu Yu** (*Academia Sinica*) ..... September 30th (Mon), 11:20–12:20

## A survey on Dieudonné modules and an arithmetic application

In this survey talk I will give a very short introduction to Dieudonné modules and the classification theorem up to isogeny due to Manin and Dieudonné. As an arithmetic application, we shall discuss a counting problem for supersingular abelian varieties using Dieudonné modules, and use this simple result to explain the classical results of Deligne, Ogus and Shioda and of Oort on superspecial abelian varieties.

**Toshiyuki Katsura** (*University of Tokyo*) ..... September 30th (Mon), 13:50–14:50

## On the supersingular locus of the moduli space of principally polarized abelian varieties in positive characteristic

In this talk, I will explain how to analyze the supersingular locus of the moduli of principally polarized abelian varieties of dimension  $g$  and the relation to the genus of quaternion hermitian form. We show that the number of irreducible components of the locus is equal to the class number of principal genus if  $g$  is odd, and to the class number of non-principal genus if  $g$  is even. This is a survey of results by Katsura–Oort and Li–Oort.

**Tomoyoshi Ibukiyama** (*Osaka University*) ..... September 30th (Mon), 15:10–16:10

## A survey on polarizations of superspecial abelian varieties and the field of definition

We generalize the well known theory of supersingular elliptic curves by Deuring and Eichler to the case of superspecial polarized abelian varieties  $(A, \lambda)$ . We explain that the number of such isomorphism classes, and the number of those which have models over  $\mathbb{F}_p$  are described by the class numbers of some quaternion hermitian lattices and some number related with  $G$ -type number that is a generalization of the usual type number. These are given by traces of Hecke operators on algebraic modular forms. Also we can count the number of irreducible components defined over  $\mathbb{F}_p$  of supersingular locus in the moduli of principally polarized supersingular abelian varieties.

More explicit theory is given when  $n = 2$ .

We also sketch an application of similar technique to the existence of maximal curves of genus 3 over  $\mathbb{F}_{p^2}$  for any odd  $p$ .

*Jiangwei Xue (Wuhan University)* ..... September 30th (Mon), 16:30–17:30

### A survey on quaternion orders and selectivity

The term “selectivity” is coined by Chinburg and Friedman in 1999, and it refers to the phenomenon that certain quadratic orders are (optimally) embeddable into some but not all types of orders within a fixed genus of quaternion orders. Such quadratic orders are “selective” in this sense. In this talk we explain the criterion for optimal selectivity, and discuss some of its applications.

*Valentijn Karemaker (Utrecht University & Stockholm University)*

..... October 1st (Tue), 10:00–11:00

### Arithmetic of twists of supersingular abelian varieties

We consider a supersingular abelian variety  $A$  defined over a finite field  $K$ , together with its  $K$ -twists. The Weil numbers of  $A$  determine  $A$  up to isogeny; the  $K$ -twists of  $A$  may have different Weil numbers. We say that  $A$  is maximal (resp. minimal) over  $K$  if all its normalised Weil numbers over  $K$  are  $-1$  (resp.  $+1$ ), since then  $A$  has a maximal (resp. minimal) number of  $K$ -points. We ask when  $A$  becomes maximal over a finite extension of  $K$ . If it does, we say  $A$  has parity 1, and if it does not, then  $A$  has parity  $-1$ . We study how twisting affects the parity and ask whether  $A$  has a  $K$ -twist with parity 1. For low-dimensional supersingular abelian varieties, we give a complete answer to this question, under conditions on their automorphism group.

This is joint work with Rachel Pries.

*Jiangwei Xue (Wuhan University)* ..... October 1st (Tue), 11:30–12:30

### On counting certain principally polarized superspecial abelian surfaces over $\mathbb{F}_p$

We study the principally polarized superspecial abelian surfaces over the prime finite field  $\mathbb{F}_p$  with Frobenius endomorphism  $\pi = \pm\sqrt{p}$ . We give a description of this set in terms of double coset spaces and give an explicit formula for its cardinality. This is a joint work with Prof. Chia-Fu Yu.

*Rachel Pries (Colorado State University)* ..... October 1st (Tue), 14:00–15:00

### Curves of high genus with many slopes of $1/2$

First, I will survey some results from a paper by Ibukiyama, Katsura, and Oort, which strongly influenced my research. Then, I will turn to the question of whether there exist smooth curves in characteristic  $p$  whose Newton polygons are far from ordinary. I will describe how clutching morphisms and PEL-type Shimura varieties can be used to give results about this question. As an application, this yields infinitely many situations when a conjecture of Oort is true and when the Newton polygon stratification of the moduli space of abelian varieties has an unlikely intersection with the Torelli locus. For example, when  $p = 2 \bmod 3$  and  $g > 1$ , we prove there exists a smooth curve of genus  $g$  in characteristic  $p$  whose Newton polygon has slopes only  $0, 1/2, 1$  and such that the multiplicity of the slope  $1/2$  is about  $2g/3$ . This is joint work with Li, Mantovan, and Tang.

*Yasuhiro Oki (University of Tokyo)* ..... October 1st (Tue), 15:30–16:30

### On certain moduli spaces of polarized supersingular QM abelian 4-folds

Let  $p$  be a prime number. In 1987, Katsura and Oort studied the structures of moduli spaces of

polarized supersingular abelian surfaces over an algebraically closed field of characteristic  $p$ . In this talk, we discuss a quaternionic analogue of their result. In the first half, we give a quick review of a connection between moduli spaces of supersingular abelian varieties and mod  $p$  reductions of certain Shimura varieties. In the second half, we give an explicit description of the structures of certain moduli spaces of polarized supersingular QM abelian 4-folds.

**Pol van Hoften** (*King's College London*) ..... October 2nd (Wed), 10:00–11:00

### A geometric Jacquet–Langlands correspondence for paramodular Siegel threefolds

It is an old idea of Serre that the classical Jacquet–Langlands correspondence between modular forms and quaternion modular forms can be realised geometrically. In this talk I will discuss an extension of these ideas to Siegel modular forms of genus two and paramodular level. We use this to prove the weight-monodromy conjecture for the Siegel threefold of paramodular level. Moreover we construct a geometric Jacquet–Langlands correspondence between  $\mathrm{GSp}_4$  and a ‘definite’ inner form, proving a conjecture of Ibukiyama.

**Jeff Achter** (*Colorado State University*) ..... October 2nd (Wed), 11:30–12:30

### Arithmetic occult periods

In a remarkable series of papers, S. Kondo discovered that several natural complex configuration spaces admit surprising uniformizations as arithmetic ball quotients, by identifying each parametrized object with the periods of some auxiliary object. In each case, the theory of canonical models of Shimura varieties gives the ball quotient the structure of a variety over the ring of integers of a cyclotomic field. In this talk, I will show that the (transcendentally-defined) period map actually respects this arithmetic structure, and thus reveal the arithmetic nature of Kondo’s period maps.

**Christophe Ritzenthaler** (*Université de Rennes 1*) ..... October 3rd (Thu), 10:00–11:00

### Reduction of curves and abelian varieties

If  $E$  is an elliptic curve with complex multiplication (CM) over a number field, it has potentially good reduction everywhere. This translates into  $j(E)$  having non negative valuation at every prime. When looking at curves of higher genus, the nature of the reduction at a prime is reflected by the valuation of the discriminant. A lot of work was done to give a closed formula in the case of CM curves of genus 2 (Goren–Lauter, Bruinier–Yang, Lauter–Viray). We will show experiments and partial results obtained in genus 3, how these questions are related to supersingular abelian varieties and the limit of the current methods. Changing then gear, we consider another natural generalization and look at Jacobians isogenous (or isomorphic) to  $E^g$ . Here, we will also present some experimental results and sketch an approach suggested by Villegas to find a closed formula when  $g = 2$ .

**Ken-ichi Sugiyama** (*Rikkyo University*) ..... October 3rd (Thu), 11:30–12:30

### On a generalization of Deuring’s results

Using the Dieudonné theory we will study a reduction of an abelian variety with complex multiplication. More precisely let  $A$  be an abelian variety of dimension  $g$  defined over a finite field  $k$  of characteristic  $p$  whose endomorphism ring contains the integer ring  $\mathcal{O}_K$  of a CM field  $K$  satisfying

$[K : \mathbb{Q}] = 2g$ . We will show that the type of  $A$  is determined by the prime decomposition of  $p$  in  $O_K$ . Our results may be regarded as a generalization of a classical theorem due to Deuring for a CM-elliptic curve. We will also discuss a sufficient condition for a prime at which the reduction of a CM-curve is maximal. This will explain several examples of curves which were found by direct computation.

**Momonari Kudo** (*Kobe City College of Technology & Kyushu University*)

..... October 3rd (Thu), 14:00–15:00

### Computational approaches to superspecial curves of genera 4 and 5 over finite fields

A superspecial curve in characteristic  $p$  is a nonsingular curve whose Jacobian (over an algebraically closed field) is the product of supersingular elliptic curves, and it is a rare case among supersingular curves. In this talk, I will present our recent works with Shushi Harashita on enumerating superspecial curves of genera 4 and 5 over finite fields. In particular, we construct enumeration algorithms, which are based on the fact by Nygaard that superspecial curves  $C$  are characterized by the Frobenius (resp. Cartier) operator acting on  $H^1(C, O_C)$  (resp.  $H^0(C, \Omega_C^1)$ ) as the zero map. I will also review and describe several computational techniques such as computation of (representation matrices of) Frobenius and Cartier operators for arbitrary algebraic curves. These computational techniques shall be useful tools to study superspecial/supersingular curves.

**Shushi Harashita** (*Yokohama National University*) ..... October 3rd (Thu), 15:30–16:30

### The existence of supersingular curves of genus 4 in arbitrary characteristic

We are interested in the existence of supersingular (nonsingular) curves of genus  $g$  in characteristic  $p$  for a given pair  $(g, p)$ . A method to construct such curves is considering fiber products of supersingular curves of smaller genera. In this talk, after reviewing previous studies using this method, I will introduce a joint work with Momonari Kudo and Hayato Senda, on the existence of supersingular curves of genus 4 in given characteristic, where we consider fiber products over  $\mathbf{P}^1$  of two supersingular elliptic curves.

**Tomoyoshi Ibukiyama** (*Osaka University*) ..... October 4th (Fri), 10:00–11:00

### Supersingular abelian surfaces, automorphisms, and the trace formulas

- (1) For surfaces, we show that the configuration of supersingular locus in the moduli is controlled by parahoric subgroups.
- (2) For a fixed supersingular abelian surface  $A$  which is not superspecial, we give an explicit formula of the number of isomorphism classes of principal polarizations  $\lambda$  of  $A$ . At the same time, we give a precise description of automorphism group  $\text{Aut}(A, \lambda)$ . The proof is based on a Mass formula of  $\text{Aut}(A, \lambda)$  and some injection of automorphism groups to a local group. By this, for any odd  $p$ , we prove a conjecture by Oort that  $\text{Aut}(A, \lambda) = \{\pm 1\}$  for generic supersingular surfaces  $A$ . We also show that  $\text{Aut}(A, \lambda)$  is never equal to  $\{\pm 1\}$  for  $p = 2$  for any principally polarized supersingular surface.

*Chia-Fu Yu (Academia Sinica)* ..... October 4th (Fri), 11:30–12:30

## Mass formula for principally polarized supersingular abelian 3-folds

The mass formula for principally polarized superspecial abelian varieties was studied and computed by Ekedahl, Ibukiyama, Katsura and Oort around 1986. In 2009, Jeng-Daw Yu and myself defined the mass for the set of principally polarized supersingular abelian varieties with a fixed isomorphism class of polarized  $p$ -divisible groups, and gave an explicit mass formula for the case  $g = 2$ . In this talk we will report work in progress on the explicit mass formula for the case where  $g = 3$ . This is joint work with Valentijn Karemaker.