Explicit combinatorial interpretation of Kerov character polynomials

(joint work with Maciej Dołęga and Valentin Féray)

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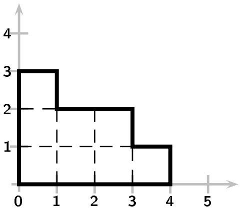
Outlook

- What can we say about the asymptotics of characters of symmetric groups $\mathfrak{S}(n)$ in the limit $n \to \infty$?
- First-order approximation is given by free cumulants.
- Free cumulants can give exact values of characters thanks to Kerov polynomials.
- The main result: explicit combinatorial interpretation of the coefficients of Kerov polynomials, in other words: very precise information about asymptotics of characters of symmetric groups.
- Open problems: relations to Schubert calculus, Toda hierarchy, ...

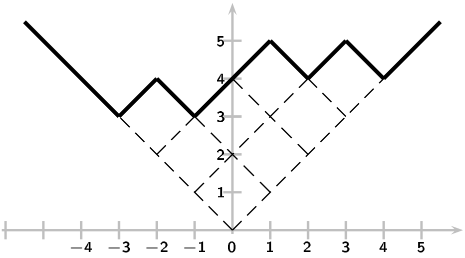
Plan

Irreducible representations of symmetric groups

Irreducible representations ρ^λ of symmetric group $\mathfrak{S}(n)$ are indexed by Young diagrams λ having n boxes.



French convention



Russian convention

Dilations of diagrams

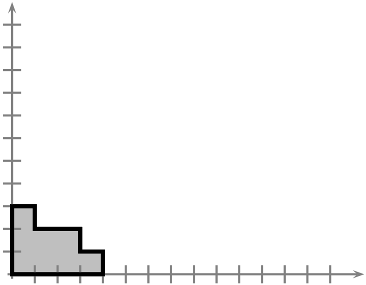
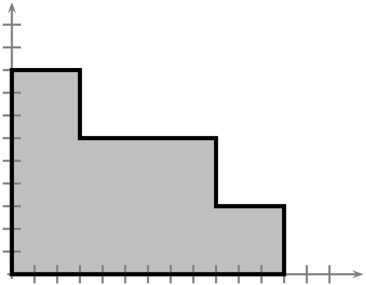


diagram λ



dilated diagram $s\lambda$ for $s = 3$

Normalized characters

We use inclusions $\mathfrak{S}(1) \subset \mathfrak{S}(2) \subset \mathfrak{S}(3) \subset \cdots$.
For $\pi \in \mathfrak{S}(k)$ and irreducible representation ρ^λ of $\mathfrak{S}(n)$ (assume $k \leq n$) we define the normalized character

$$\Sigma_\pi^\lambda = \underbrace{n(n-1)\cdots(n-k+1)}_{k \text{ factors}} \frac{\operatorname{Tr} \rho^\lambda(\pi)}{\text{dimension of } \rho^\lambda}.$$

Most interesting case: characters on cycles

$$\Sigma_k^\lambda = \Sigma_{(1,2,\dots,k)}^\lambda.$$

Problem

For fixed $k \geq 1$ what can we say about $\Sigma_k^{s\lambda}$ for $s \to \infty$?

Free cumulants

We define free cumulants $R_2^\lambda, R_3^\lambda, \dots$ of diagram λ to be asymptotically the dominant terms of the character on cycles:

$$R_k^\lambda = \lim_{s\to\infty} \frac{1}{s^k} \Sigma_{k-1}^{s\lambda}.$$

Free cumulants are homogeneous with respect to dilations: $R_k^{s\lambda} = s^k R_k^\lambda$.

First order approximation: $\Sigma_{k-1} \approx R_k$.

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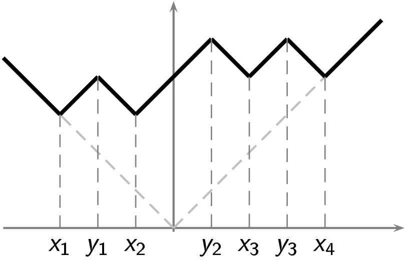
Free cumulants are very very nice quantities to describe a Young diagram: they can be explicitly calculated in several approaches (next transparencies)
and are very useful in asymptotic representation theory.

Cauchy transform and free cumulants



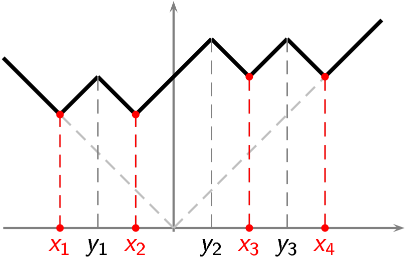
Cauchy transform of λ:

$$G(z) = \frac{(z-y_1)(z-y_2)\cdots}{(z-x_1)(z-x_2)\cdots}$$

$$\Sigma_{k-1} = \frac{-1}{k-1}[z^{-1}]_\infty \frac{1}{G(z-1)G(z-2)\cdots G(z-k+1)}$$

$$R_k = \frac{-1}{k-1}[z^{-1}]_\infty \left(\frac{1}{G(z)}\right)^{k-1}$$

Transition measure



Cauchy transform of λ:

$$G(z) = \frac{(z - y_1)(z - y_2)\cdots}{(z - x_1)(z - x_2)\cdots}$$

Kerov's transition measure μ^λ is a probability measure on $\mathbb{R}$ such that:

$$G(z) = \int \frac{1}{z - x} d\mu^\lambda(x)$$

Related to induced representation $\rho^\lambda \uparrow_{\mathfrak{S}(n)}^{\mathfrak{S}(n+1)}$

Moments of transition measure

Moments $M_2, M_3, \dots$ of transition measure:

$$G(z) = \frac{1}{z} + \frac{M_2}{z^2} + \frac{M_3}{z^3} + \cdots$$

$$M_k = \int x^k \, d\mu^\lambda$$

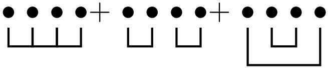
are also moments of Jucys-Murphy element
$(1\star) + (2\star) + \cdots + (n\star) \in \mathbb{C}[\mathfrak{S}(n+1)]$, where $\star = n+1$:

$$M_k = \rho^\lambda \left(\left[(1\star) + (2\star) + \cdots + (n\star) \right]^k \downarrow^{\mathfrak{S}(n+1)}_{\mathfrak{S}(n)} \right)$$

Free cumulants

Relation between moments $M_2, M_3, \dots$ and free cumulants $R_2, R_3, \dots$:

$$M_k = \sum_{\substack{\pi: \\ \text{non-crossing} \\ \text{partitions} \\ \text{of } \{1,\dots,k\}}} R_\pi$$

Example: $M_4 =$  $= R_4 + R_2^2 + R_2^2$

- free cumulants come from Voiculescu's free probability theory
- describe also asymptotics of random matrices

Plan

Kerov polynomials

Free cumulants give approximations of characters:

$$\Sigma_k \approx R_{k+1},$$

but they can also give exact values of characters thanks to Kerov character polynomials:

$$\begin{aligned}
\Sigma_1 &= R_2, \\
\Sigma_2 &= R_3, \\
\Sigma_3 &= R_4 + R_2, \\
\Sigma_4 &= R_5 + 5R_3, \\
\Sigma_5 &= R_6 + 15R_4 + 5R_2^2 + 8R_2, \\
\Sigma_6 &= R_7 + 35R_5 + 35R_3R_2 + 84R_3.
\end{aligned}$$

Kerov conjecture

Theorem/Conjecture (Kerov)

For each $k \geq 1$ there exists a universal polynomial $K_k(R_2, R_3, \dots)$ with non-negative integer coefficients called Kerov character polynomial such that

$$\Sigma_k = K_k(R_2, R_3, \dots)$$

What is the combinatorial interpretation of coefficients?

Féray: Kerov's conjecture is true, coefficients have a complicated combinatorial interpretation.

Main result of this talk: explicit combinatorial interpretation of coefficients.

Linear terms of Kerov polynomials

Theorem (Biane and Stanley)

The coefficient $[R_l]K_k$ is equal to the number of pairs (σ_1, σ_2) where

- $\sigma_1, \sigma_2 \in \mathfrak{S}(k)$ *are such that* $\sigma_1 \circ \sigma_2 = (1, 2, \dots, k)$,
- σ_2 *consists of one cycle,*
- σ_1 *consists of* $l - 1$ *cycles.*

Quadratic terms of Kerov polynomials

For a permutation π we denote by $C(\pi)$ the set of cycles of π.

Theorem (Féray)

The coefficient $[R_{l_1} R_{l_2}]K_k$ is equal to the number of triples (σ_1, σ_2, q) with the following properties:

- $\sigma_1, \sigma_2 \in \mathfrak{S}(k)$ *are such that* $\sigma_1 \circ \sigma_2 = (1, 2, \dots, k)$*;*
- σ_2 *consists of* 2 *cycles;*
- σ_1 *consists of* $l_1 + l_2 - 2$ *cycles;*
- $q : C(\sigma_2) \to \{l_1, l_2\}$ *is a surjective map on cycles of* σ_2*;*
- *for each cycle* c *of* σ_2 *there are at least* $q(c)$ *cycles of* σ_1 *which intersect nontrivially* c*.*

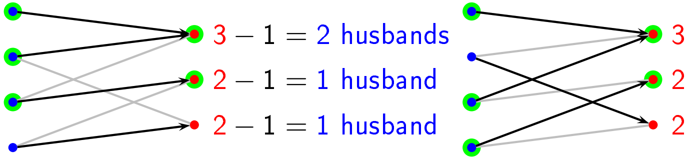
The main result: combinatorial interpretation of Kerov polynomials

Theorem

The coefficient $[R_2^{s_2} R_3^{s_3} \cdots] K_k$ *is equal to the number of triples* (σ_1, σ_2, q) *such that*

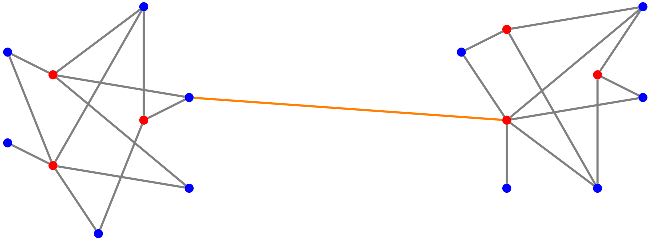
- $\sigma_1, \sigma_2 \in \mathfrak{S}(k)$ *are such that* $\sigma_1 \circ \sigma_2 = (1, 2, \dots, k)$;
- $|C(\sigma_2)| = s_2 + s_3 + \cdots$;
- $|C(\sigma_1)| + |C(\sigma_2)| = 2s_2 + 3s_3 + 4s_4 + \cdots$;
- $q : C(\sigma_2) \to \{2, 3, \dots\}$ *is a coloring such that each color* $i \in \{2, 3, \dots\}$ *is used* s_i *times;*
- *for every nontrivial set* $\emptyset \subsetneq A \subsetneq C(\sigma_2)$ *of cycles of* σ_2 *there are more than* $\sum_{i \in A} \big(q(i) - 1\big)$ *cycles of* σ_1 *which intersect* $\bigcup A$.

Marriage interpretation



Example: coefficient $[R_2^2 R_3]K_k$. We consider a bipartite graph $\mathcal{V}_{\sigma_1,\sigma_2}$ with the vertices corresponding to cycles of σ_1 (boys) and cycles of σ_2 (girls). We draw an edge if two cycles intersect (boy is allowed to marry a girl). Each boy wants to marry one girl and each girl $g \in C(\sigma_2)$ wants to marry $q(g) - 1$ boys.
We require that it is possible to arrange marriages and that for each non-trivial set of girls the set of their husbands is not uniquely determined.

Restriction on graphs



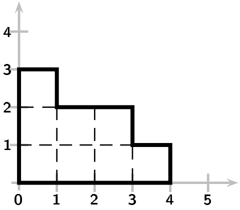
Corollary

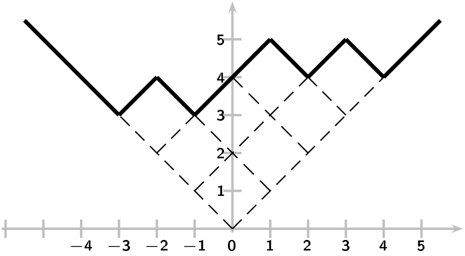
If there exists an disconnecting edge with at least one girl in both components then the factorization cannot contribute (no matter which labeling we choose).

Plan

1. Free cumulants of Young diagrams
2. Kerov character polynomials
3. Proof of Kerov conjecture
   - Fundamental functionals $S_2, S_3, \dots$ of shape
   - Stanley polynomials
   - Toy example: quadratic terms of Kerov polynomials
4. Applications and open problems

Fundamental functionals $S_2, S_3, \dots$ of shape





$\text{contents}_{(x,y)} = x - y$ $\qquad$ $\text{contents}_{(x,y)} = x$

Fundamental functionals of shape of λ:

$$S_n^\lambda = (n-1) \iint_{\square \in \lambda} (\text{contents}_\square)^{n-2} \, d\square$$

Fundamental functionals $S_2, S_3, \dots$ of shape

There are explicit formulas which express functionals $S_2, S_3, \dots$ in terms of free cumulants $R_2, R_3, \dots$ and conversely.

$$S_n = \sum_{l \geq 1} \frac{1}{l!} (n-1)_{l-1} \sum_{\substack{k_1, \dots, k_l \geq 2 \\ k_1 + \cdots + k_l = n}} R_{k_1} \cdots R_{k_l},$$

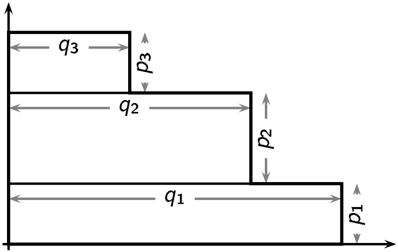
$$R_n = \sum_{l \geq 1} \frac{1}{l!} (-n+1)^{l-1} \sum_{\substack{k_1, \dots, k_l \geq 2 \\ k_1 + \cdots + k_l = n}} S_{k_1} \cdots S_{k_l},$$

Example:

$$\frac{\partial^2}{\partial R_{k_1} \partial R_{k_2}} \mathcal{F} = \frac{\partial^2}{\partial S_{k_1} \partial S_{k_2}} \mathcal{F} + (k_1 + k_2 - 1) \frac{\partial}{\partial S_{k_1 + k_2}} \mathcal{F}.$$

All derivatives at $R_2 = R_3 = \cdots = S_2 = S_3 = \cdots = 0$.

Stanley polynomials



For numbers $p_1, p_2, \dots, q_1, q_2, \dots$ we consider multirectangular (generalized) Young diagram $\mathbf{p} \times \mathbf{q}$.

Theorem (conjectured by Stanley, proved by Féray)

For any permutation π the normalized character $\Sigma_\pi^{\mathbf{p}\times\mathbf{q}}$ is a polynomial in $p_1, p_2, \dots, q_1, q_2, \dots$, called Stanley polynomial, for which there is an explicit formula.

Stanley-Féray character formula

Theorem (conjectured by Stanley, proved by Féray)

For $\pi \in \mathfrak{S}(n)$

$$\Sigma_{\pi}^{\mathbf{p}\times\mathbf{q}} = \sum_{\substack{\sigma_1,\sigma_2\in\mathfrak{S}(n)\\ \sigma_1\circ\sigma_2=\pi}} \sum_{\phi_2 : C(\sigma_2)\to\mathbb{N}} (-1)^{\sigma_1} \left[\prod_{b\in C(\sigma_1)} q_{\phi_1(b)} \prod_{c\in C(\sigma_2)} p_{\phi_2(c)} \right],$$

where coloring $\phi_1 : C(\sigma_1) \to \mathbb{N}$ *is defined by*

$$\phi_1(c) = \max_{\substack{b\in C(\sigma_2),\\ b \text{ and } c \text{ intersect}}} \phi_2(b) \qquad \text{for } c \in C(\sigma_1)$$

The Stanley polynomial depends on the graph $\mathcal{V}_{\sigma_1,\sigma_2}$.

Stanley-Féray character formula, toy version

Corollary

For $\pi \in \mathfrak{S}(n)$

$$(-1)[p_1 p_2 q_1^i q_2^j]\Sigma_\pi^{\mathbf{p}\times\mathbf{q}}$$

is equal to the number of factorizations $\pi = \sigma_1 \circ \sigma_2$ *such that* σ_1 *has* $i + j$ *cycles,* $\sigma_2 = \{c_1, c_2\}$ *has two (labeled) cycles and such that there are* j *cycles of* σ_1 *which intersect* c_2*.*

Stanley polynomials and functionals $S_2, S_3, \dots$

Theorem

If $\mathcal{F}$ is a sufficienly nice function on the set of generalized Young diagrams then it as a polynomial in $S_2, S_3, \dots$.

$$\frac{\partial}{\partial S_{k_1}} \cdots \frac{\partial}{\partial S_{k_l}} \mathcal{F} \bigg|_{S_2 = S_3 = \cdots = 0} = [p_1 q_1^{k_1 - 1} \cdots p_l q_l^{k_l - 1}] \mathcal{F}^{\mathbf{p} \times \mathbf{q}}$$

- Therefore expansion of Σ_π in terms of $S_2, S_3, \dots$ can be extracted from Stanley polynomials.
- Stanley polynomials are explicitly given by Stanley-Féray formula and depend on geometry of bipartite graphs $\mathcal{V}$.
- Once we know the expansion of Σ_π in terms of $S_2, S_3, \dots$ we can find expansion of Σ_π in terms of free cumulants $R_2, R_3, \dots$.

Free cumulants vs fundamental functionals

Free cumulants $R_2, R_3, \ldots$

- describe Young diagram in language of representation theory
- best quantities for calculating characters

Functionals $S_2, S_3, \ldots$

- describe Young diagram in language of its shape
- directly related to Stanley polynomials

Toy example: $[R_{k_1} R_{k_2}]\Sigma_n$

$$\frac{\partial^2}{\partial R_{k_1} \partial R_{k_2}} \Sigma_n = \frac{\partial^2}{\partial S_{k_1} \partial S_{k_2}} \Sigma_n + (k_1 + k_2 - 1) \frac{\partial}{\partial S_{k_1+k_2}} \Sigma_n =$$

$$[p_1 p_2 q_1^{k_1-1} q_2^{k_2-1}] \Sigma_n^{\mathbf{p}\times\mathbf{q}} + (k_1 + k_2 - 1)[p_1 q_1^{k_1+k_2-1}] \Sigma_n^{\mathbf{p}\times\mathbf{q}} =$$

$$[p_1 p_2 q_1^{k_1-1} q_2^{k_2-1}] \Sigma_n^{\mathbf{p}\times\mathbf{q}} - [p_1 p_2 q_2^{k_1+k_2-2}] \Sigma_n^{\mathbf{p}\times\mathbf{q}}$$

Toy example: $[R_{k_1} R_{k_2}]\Sigma_n$

We are interested in factorizations $\sigma_1 \circ \sigma_2 = (1, \dots, n)$ such that σ_1 has $k_1 + k_2 - 2$ cycles and $\sigma_2 = \{c_1, c_2\}$ has two cycles.

#(fact. such that c_1 has $\geq k_1$ friends, c_2 has $\geq k_2$ friends) $=$

#(all fact.) $-$ #(fact. such that c_1 has $\leq k_1 - 1$ friends)

$-$#(fact. such that c_2 has $\leq k_2 - 1$ friends) $=$

$$(-1) \sum_{\substack{i+j=k_1+k_2-2,\\ 1\leq j}} \left[p_1 p_2 q_1^i q_2^j\right] \Sigma_k^{\mathbf{p}\times\mathbf{q}} + \sum_{\substack{i+j=k_1+k_2-2,\\ 1\leq i\leq k_1-1}} \left[p_1 p_2 q_1^j q_2^i\right] \Sigma_k^{\mathbf{p}\times\mathbf{q}}$$

$$+ \sum_{\substack{i+j=k_1+k_2-2,\\ 1\leq j\leq k_2-1}} \left[p_1 p_2 q_1^i q_2^j\right] \Sigma_k^{\mathbf{p}\times\mathbf{q}} =$$

$$[p_1 p_2 q_1^{k_1-1} q_2^{k_2-1}]\Sigma_n^{\mathbf{p}\times\mathbf{q}} - [p_1 p_2 q_2^{k_1+k_2-2}]\Sigma_n^{\mathbf{p}\times\mathbf{q}}$$

Plan

Applications

- coefficients of Kerov polynomials are small,
- Kerov polynomials give characters as simple sums without too many cancellations,
- optimal estimates for characters

Exotic interpretations of Kerov polynomials

Conjecture

Maybe coefficients of Kerov polynomials

- *are equal to dimensions of some intersection (co)homologies of Schubert varieties? [conjecture of Philippe Biane]*
- *are equal to something related to moduli space of analytic maps on Riemann surfaces? or ramified coverings of a sphere? [conjecture of Śniady]*
- *are algebraic solutions to some integrable hierarchy (Toda?) and their coefficients are related to the tau function of the hierarchy? [conjecture of Jonathan Novak]*

Open problems

- is there some analogue of Kerov character polynomials for the representation for unitary groups $U(d)$?
- do Kerov polynomials for $\mathfrak{S}(n)$ tell us someting about representations of the unitary groups $U(d)$?
- is there some analogue of Kerov character polynomials in the random matrix theory?
- is it possible to study Kerov polynomials in such a scaling that phenomena of universality of random matrices occur?

Bibliography

Valentin Féray, Maciej Dołęga, Piotr Śniady.
Explicit combinatorial interpretation of Kerov character polynomials as numbers of permutation factorizations
Preprint 2008