Garsia-Haiman modules for hooks and its graded characters at roots of unity

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GARSIA-HAIMAN MODULES

LET

- S_n : the symmetric group of n letters,
- $\mu = (\mu_i)$: a partition of n,
- D_μ : "Garsia-Haiman module (corr. to μ)"

FACTS

- $D_\mu = \bigoplus_{r=0}^{n(\mu)} \bigoplus_{s=0}^{n(\mu')} D_\mu^{r,s}$: doubly graded S_n-modules,
  $n(\mu) = \sum_i (i-1)\mu_i$,
- $\dim D_\mu = n!$ (Haiman's $n!$ theorem),
- $D_\mu^{*,s} := \bigoplus_{r=0}^{n(\mu)} D_\mu^{r,s}, \quad s = 0, 1, \ldots, n(\mu')$,
- $D_\mu^{*,0} \cong R_\mu$ (Springer module), as graded S_n-modules.

REMARKS

- $R_\mu = H^*(X_\mu)$ with graded S_n-module structure,
- $\tilde{K}_{\lambda\mu}(q,t) = \sum_{r,s} [D_\mu^{r,s} : L^\lambda] q^s t^r \in \mathbf{Z}_{\geq 0}[q,t]$.

CONJECTURE

LET

- l : a positive integer,
- $D_\mu^{*,s}(k;l) := \bigoplus_{r\equiv k \mod l} D_\mu^{r,s}, \quad k = 0,1,\ldots,l-1,$
- $M_\mu := \max\{m_i \mid i \geq 1\}$, $\mu = (i^{m_i})$.

<u>CONJECTURE</u> For each $s = 0,1,\ldots,n(\mu')$,

$$1 \leq l \leq M_\mu \Longrightarrow \dim D_\mu^{*,s}(k;l) = \frac{1}{l}\dim D_\mu^{*,s}.$$

<u>REMARK :</u> <u>$s = 0$</u>

- $D_\mu^{*,0} = R_\mu$: Springer module,
- ${}^\forall l$: a positive integer, $1 \leq l \leq M_\mu$,
- ${}^\exists S_n(l)$: a subgroup of S_n,
- ${}^\exists Z_\mu(k;l)$: $S_n(l)$-modules of equal dimension,
- $D_\mu^{*,0}(k;l) \cong_{S_n} \mathrm{Ind}_{S_n(l)}^{S_n} Z_\mu(k;l), \quad k = 0,1,\ldots,l-1.$
- $\dim D_\mu^{*,0}(k;l) = \dfrac{1}{l}\dim D_\mu^{*,0}, \quad k = 0,1,\ldots,l-1.$
  $(\dim \mathrm{Ind}_{S_n(l)}^{S_n} Z_\mu(k;l) = [S_n : S_n(l)]\dim Z_\mu(k;l))$

EXAMPLE

- $\mu = (3, 1, 1)$,
- Dimensions of $D_\mu^{r,s}$:

	0	1	2	3
0	1	4	9	6
1	4	11	16	9
2	9	16	11	4
3	6	9	4	1

- Irreducible decomposition of $D_\mu^{r,s}$:

	0	1	2	3
0	5	41	$41,\ 32$	31^2
1	41	$32,\ 31^2$	$32,\ 31^2,\ 2^21$	$2^21,\ 21^3$
2	$41,\ 32$	$32,\ 31^2,\ 2^21$	$31^2,\ 2^21$	21^3
3	31^2	$2^21,\ 21^3$	21^3	1^5

n! theorem

$$\to \ \tilde{K}_{\lambda\mu}(q,t) = \sum_{r,s} [D_\mu^{r,s} : L^\lambda] q^s t^r$$

$\to$ Figure of Macdonald's book

PROBLEM AND RESULT

LET

- μ : a partition of n,
- l : a positive integer, $1 \le l \le M_\mu$,
- $s = 0, 1, \ldots, n(\mu')$.

PROBLEM

Find

- $S_n(l)$: a subgroup of S_n,
- $Z^{*,s}_\mu(k;l)$: $S_n(l)$-modules of equal dimension $(0 \le k \le l-1)$,

such that

$$D^{*,s}_\mu(k;l) \cong_{S_n} \mathrm{Ind}^{S_n}_{S_n(l)} Z^{*,s}_\mu(k;l), \ 0 \le k \le l-1$$

MAIN RESULT(M, 2008, [M])

If μ is a hook, the problem is solved :

$$\dim D^{*,s}_\mu(k;l) = \frac{1}{l} \dim D^{*,s}_\mu, \quad 0 \le k \le l-1$$

EXAMPLE

- $\mu = (3,1,1)$ $(M_\mu = 2)$,
- $l = 2$,
- $T = \begin{array}{|c|c|c|}\hline 3 & 4 & 5 \\ \hline 2 \\ \cline{1-1} 1 \\ \cline{1-1}\end{array} \xrightarrow{\text{mod } l} \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array} + \begin{array}{|c|c|c|}\hline 3 & 4 & 5 \\ \hline\end{array}$
- $a := (12)$, $\hat{\mu} := \begin{array}{|c|c|c|}\hline & & \\ \hline\end{array}$,
- $S_n(l) := (S_{\{1\}} \times S_{\{2\}}) \rtimes \langle a \rangle \times S_{\{3,4,5\}} \cong C_2 \times S_3$,
- $\varphi_l^{(m)} \; : \; \langle a \rangle \longrightarrow \mathbf{C}^\times \; : \; a \longmapsto \zeta_l^m$, $\zeta_l = e^{\frac{2\pi\sqrt{-1}}{l}}$.
- $\mathbf{C}_l^{(m)}$: representation space of $\varphi_l^{(m))}$,
- $Z_\mu^{*,s}(k:l) := \bigoplus_{r=0}^{n(\hat{\mu})} \mathbf{C}_l^{(k-r)} \otimes D_{\hat{\mu}}^{r,s}, \quad 0 \le k \le l-1$
- $\dim Z_\mu^{*,s}(k;l) = \dim D_{\hat{\mu}}^{*,s}, \quad \forall k$,
- $D_\mu^{*,s}(k;l) \cong \mathrm{Ind}_{S_n(l)}^{S_n} Z_\mu^{*,s}(k;l), \quad \forall k$:
  - $\dim D_\mu^{*,1}(k;2) = 20$,
  - $\dim D_{\hat{\mu}}^{*,1} = 2$,
  - $[S_n : S_n(l)] = 5!/2 \cdot 3! = 10$.

SKETCH OF PROOF

$$D_\mu^{*,s}(k;l) \cong_{S_n} \mathrm{Ind}_{S_n(l)}^{S_n} Z_\mu^{*,s}(k;l), \quad k = 0, 1, \ldots, l-1$$

$$\Leftrightarrow D_\mu^{*,s} \cong_{S_n \times C_l} \mathrm{Ind}_{S_r}^{S_n} D_{\hat{\mu}}^{*,s} \quad (s = 0, 1, \ldots, n(\mu'))$$

- Compare the character values for $(w, a^j) \in S_n \times C_l$,
- Unite character identities with respect to s.

Then we can see that it is enough to show:

$$\tilde{X}_\rho^\mu(q, \zeta_l^j) = \sharp\{\sigma \in S_n/S_r | w\sigma a^j \equiv \sigma \bmod S_r\} \tilde{X}_{\rho(\tau)}^{\hat{\mu}}(q, \zeta_l^j), \quad (*)$$

where

- $\tilde{X}_{\rho(w)}^\mu(q,t) := \sum_{r=0}^{n(\mu)} \sum_{s=0}^{n(\mu')} t^r q^s \mathrm{char} D_\mu^{r,s}(w),$
- $\tau \in S_r$ is defined by $w\sigma a^j = \sigma\tau$.

Finally:

"Factorization formula"+ "Plethystic formula" for $\tilde{H}_\mu(x;q,t) \Rightarrow (*)$

MODIFIED MACDONALD POLYNOMIALS AT ROOTS OF UNITY

$\tilde{H}_\mu(x;q,t)$: "modified" Macdonald polynomial.

PROPOSITION(Descouens-M, 2007, [DM])
$\mu = (i^{m_i})$ with $m_r \geq l$ for some r.

1) $\tilde{H}_\mu(x;q,\zeta_l) = \tilde{H}_{(r^l)}(x;q,\zeta_l)\tilde{H}_{\mu\setminus(r^l)}(x;q,\zeta_l)$,

2) $\tilde{H}_{(r^l)}(x;q,\zeta_l) = \left\{\prod_{i=1}^{r}(1-q^{il})\right\}(p_l \circ h_r)\left(\frac{x}{1-q}\right)$,

$$(p_l \circ h_r)\left(\frac{x}{1-q}\right) = \sum_{\lambda\vdash r} z_\lambda^{-1}\frac{p_{l\lambda}(x)}{(1-q)^{l\lambda}}.$$

Haiman's $n!$ theorem implies :

PROPOSITION μ, $\rho \vdash n$,

$$\tilde{X}^\mu_\rho(q,t) = \langle \tilde{H}_\mu(x;q,t), p_\rho(x)\rangle.$$

REMARK $\tilde{X}^\mu_\rho(0,t) = Q^\mu_\rho(t)$: Green polynomial.

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