Wreath product generalization of the Gelfand triple (S_{2n}, H_n, ξ)

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1 Gelfand Triple

G : Finite group

H : Subgroup of G

ξ : Representation of $H/\mathbb{C}$

Definition 1.1.

(G, H, ξ) : Gelfand triple $\iff \xi \uparrow_H^G$ is multiplicity-free.

$((G, H)$: Gelfand pair $\iff 1 \uparrow_H^G$ is multiplicity-free.)

In the below, let ξ be a 1-dim. repn. of H.

Assume (G, H, ξ) is a Gelfand triple. Put

$$\xi \uparrow_H^G = \bigoplus_{i=1}^{t} V_i, \quad (V_i : \text{a unitary irred. repn. of } G)$$

Then

$$^{\exists_1} v(i) \in V_i \;\text{ s. t. } hv(i) = \xi(h)v(i) \; (h \in H) \text{ and } |v(i)| = 1.$$

Define **ξ-spherical function** **by**

$$\omega_i(g) = \langle v(i), gv(i) \rangle.$$

$$\big(\omega_i(hg) = \omega_i(gh) = \xi(h^{-1})\omega_i(g), \; (h \in H)\big)$$

2 Hecke algebra

$G \supset H$, ξ: 1-dim repn. of H.

$$\mathbb{C}G\text{: Group algebra of } G \overset{identify}{\longleftrightarrow} \{f : G \longrightarrow \mathbb{C}\}$$

$$\sum_{g \in G} f(g)g \longleftrightarrow f : g \mapsto f(g)$$

- $$e_\xi = \frac{1}{|H|} \sum_{h \in H} \xi(h^{-1})h$$

Hecke algebra : $\mathcal{H}^\xi(G, H) := e_\xi \mathbb{C}G e_\xi \subset \mathbb{C}G$.

$$\mathcal{H}^\xi(G, H) = \{f : G \longrightarrow \mathbb{C} \mid f(hg) = f(gh) = \xi(h^{-1})f(g),\ h \in H\}$$

Proposition 2.1.

$\mathcal{H}^\xi(G,H)$ is commutative $\Leftrightarrow$ (G,H,ξ) is a Gelfand triple.

Proposition 2.2.

$\{\omega_1,\cdots,\omega_s\}$ is an orthonormal basis of $\mathcal{H}^\xi(G,H)$.

Inner product of $\mathcal{H}^\xi(G,H)$;

$$\langle f,f'\rangle_G=\frac{1}{|G|}\sum_{g\in G}f(g)\overline{f'(g)}.$$

- $\{g_1,\cdots,g_t\}(t\geq s)$: a complete representatives of $H\backslash G/H$.

Then $\{e_\xi g_i e_\xi \mid \ 1\leq i\leq t,\ \ e_\xi g_i e_\xi\neq 0\}$ is a basis of $\mathcal{H}^\xi(G,H)$.

3 $G = S_{2n}$ and $H = H_n$

- S_{2n} : Symmetric group
- H_n : Hyperoctahedral group $(\cong \mathbb{Z}/2\mathbb{Z} \wr S_n)$

$$= \langle \underbrace{(2i-1, 2i)}_{(\mathbb{Z}/2\mathbb{Z})^n-part}, \underbrace{(2j-1, 2j+1)(2j, 2j+2)}_{S_n-part}; 1 \leq i \leq n, 1 \leq j \leq n-1 \rangle.$$

<u>1-dim. repn. of H_n</u>: $1, \delta, \varepsilon, \delta \otimes \varepsilon$

$$\begin{cases} \delta((2j-1, 2j+1)(2j, 2j+2)) = \varepsilon((2i-1, 2i)) = 1 \\ \delta((2i-1, 2i)) = \varepsilon((2j-1, 2j+1)(2j, 2j+2)) = -1 \end{cases}$$

Theorem 3.1. **(A. James)**

$(S_{2n}, H_n, 1)$ **is a Gelfand triple.** $(1 \uparrow_{H_n}^{S_{2n}} = \bigoplus_{\lambda \vdash n} S^{2\lambda}.)$

"**1-Spherical fn.** $\omega^{\lambda} \mapsto z_{\lambda} = |H_n| \sum_{\rho \vdash n} \frac{1}{z_{\rho}} \omega_{\rho}^{\lambda} p_{\rho}$ **(zonal poly.)**"

$2\lambda = (2\lambda_1, 2\lambda_2, \cdots).$

Theorem 3.2. **(J. Stembridge)**

$(S_{2n}, H_n, \varepsilon)$ **is a Gelfand triple.** $(\varepsilon \uparrow_{H_n}^{S_{2n}} = \bigoplus_{\lambda \vdash n, strict} S^{D(\lambda)}.)$

" ε**-spherical fn.** $\xi^{\lambda} \mapsto Q_{\lambda} \propto \sum_{\rho \in OP_n} \frac{1}{z_{\rho}} \xi_{\rho}^{\lambda} p_{\rho}$ **(Schur's** Q**-fn).**"

$D(\lambda) = (\lambda_1, \lambda_2, \cdots | \lambda_1 - 1, \lambda_2 - 1, \cdots)$; **in Frobenius notation**

- $SG_{2n} := G \wr S_{2n}$
- $HG_n := \{(g_1, g_1, g_2, g_2, \cdots, g_n, g_n; \sigma); g_i \in G, \sigma \in H_n\}$

Theorem 3.3. [M]

$(SG_{2n}, HG_n, 1)$ is a Gelfand triple.

" 1-spherical fn. $\mapsto$ product of zonal and Schur polys."

Problem 3.4. Let Θ be a 1-dim. repn. of HG_n.

$\longrightarrow$ (SG_{2n}, HG_n, Θ) : Gelfand triple?

$\longrightarrow$ $\Theta \uparrow_{HG_n}^{SG_{2n}} = ?$

$\longrightarrow$ When $e_\Theta x e_\Theta = 0$?

$\longrightarrow$ Compute Θ-Spherical functions (Evaluate them on each double cosets).

4 (SG_{2n}, HG_n, Θ)

4.1 1-dim. repn. of HG_n

Remark

$$HG_n \cong (G \times S_2) \wr S_n.$$

Let ξ be an 1-dim. repn. of G.

1-dim. repn. of HG_n:

$$\Theta_{\xi,\pi}(g_1, g_1, \cdots, g_n, g_n; \sigma) = \xi(g_1 g_2 \cdots g_n)\pi(\sigma),$$

where π is a 1-dim. repn. of H_n (one of 4).

4.2 Double Coset

For $x = (g_1, g_2, \cdots, g_{2n}; \sigma) \in SG_{2n}$, Define a graph

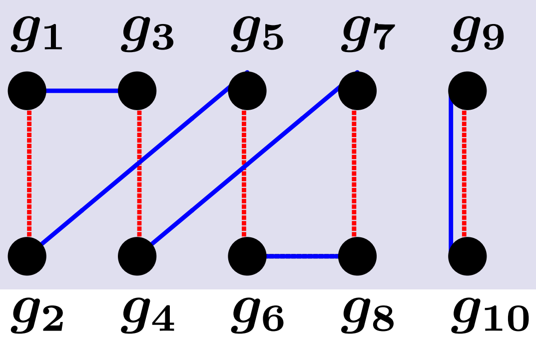
$$\Gamma_x = (\underbrace{V_x}_{\text{vertices}}, \underbrace{E_x}_{\text{edges}}) \text{ by}$$

$$\begin{cases} V_x = \{g_1, g_2, \cdots, g_{2n}\}, \\ E_x = \{\{g_{2i-1}, g_{2i}\}, \{g_{\sigma(2i-1)}, g_{\sigma(2i)}\}; 1 \leq i \leq n\}. \end{cases}$$

Example 4.1.

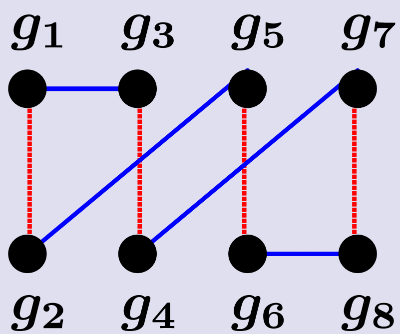
$x = (g_1, g_2, \cdots, g_{10}; (1,4,3)(2,7)(5,8)(9,10))$

$\Gamma_x =$



Definition 4.2.

- L : a circuit (=connected component of Γ_x) of Γ_x

$L =$ 

$\longrightarrow$ $$p(L) = g_1^{-1} g_2 g_5^{-1} g_6 g_8^{-1} g_7 g_4^{-1} g_3$$

- $p(L)$: circuit product of length 4 (= $\sharp$ of red line=: $|L|$).

G_* : the set of conjugacy class of G

$$G_{**} = \{C \cup C^{-1} \; ; \; C \in G_*\}, \qquad (C^{-1} = \{g^{-1} \mid g \in C\}).$$

Example 4.3.

$G = \mathbb{Z}/6\mathbb{Z} = \{1, a, a^2, a^3, a^4, a^5\}$

$G_{**} = \{\{1\}, \{a, a^5\}, \{a^2, a^4\}, \{a^3\}\}$

Definition 4.4.

$x \in SG_{2n}$, $R \in G_{**}$

$m_k^x(R) = |\{L \; : \; \text{circuit of } \Gamma_x \mid p(L) \in R, |L| = k\}|$,

$\rho^x(R) := 1^{m_1^x(R)} 2^{m_2^x(R)} \ldots$

$CT(x) := (\rho^x(R); R \in G_{**})$: circuit type of x

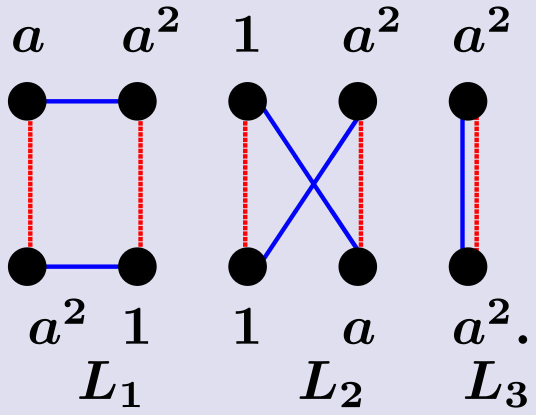
Example 4.5.

$G = \mathbb{Z}/3\mathbb{Z} = \{1, a, a^2\}$

$G_{**} = \{R_1 = \{1\}, R_2 = \{a, a^2\}\}$

$x = (a, a^2, a^2, 1, 1, 1, a^2, a, a^2, a^2; (23)(5678)(9, 10))$

$\Gamma_x =$



$$\begin{cases} p(L_1) = a(a^2)^{-1}1(a^2)^{-1} = 1 & (|L_1| = 2) \\ p(L_2) = 1(1)^{-1}a^2a^{-1} = a & (|L_2| = 2) \\ p(L_3) = a^2(a^2)^{-1} = 1 & (|L_3| = 1) \end{cases}$$

$$CT(x) = (\rho^x(R_1), \rho^x(R_2)) = ((2, 1), (2)).$$

Theorem 4.6.

$x, y \in SG_{2n}$.

$CT(x) = CT(y) \Leftrightarrow x \in HG_n y HG_n$.

Remark 4.7.

$HG_n \backslash SG_{2n} / HG_n \leftrightarrow \{\underline{\rho}\ ;\ |G_{**}|-\text{tuple of partitions}\ , |\underline{\rho}| = n\}$

4.3 Representation Theory of wreath product

$G^* = \{\gamma_1, \gamma_2, \cdots, \gamma_r\}$: the set of Irred. repns of G,
$S_n^* = \{S^\lambda \mid |\lambda| = n\}$

For $\gamma \in G^*$, we can define an action of $G \wr S_n$ on $\gamma^{\otimes n} \otimes S^\lambda$ by

$$(g_1, \cdots, g_n; \sigma) v_1 \otimes \cdots \otimes v_n \otimes w = g_1 v_{\sigma^{-1}(1)} \otimes \cdots \otimes g_n v_{\sigma^{-1}(n)} \otimes \sigma w.$$

For $\underline{n} = (n_i | 1 \le i \le r) \in \mathbb{N}_0^r$ and $|\underline{n}| = \sum_{i=1}^{r} n_i = n$ (i.e. r-composition of n), we define a subgroup of $G \wr S_n$ by

$$G \wr S_{\underline{n}} := \prod_{i=1}^{r} G \wr S_{n_i} \quad .$$

Proposition 4.8.
$$(G \wr S_n)^* = \{\bigotimes_{i=1}^{r} \gamma^{\otimes n_i} \otimes S^{\lambda^i} \uparrow_{G \wr S_{\underline{n}}}^{G \wr S_n} \mid |\underline{n}| = n, \ |\lambda^i| = n_i\}.$$

4.4 Analysis of (SG_2, HG_1)

- ξ : 1-dim. repn. of G
- $\hat{\xi} : (g, g; \sigma) \mapsto \xi(g)$

Proposition 4.9. $(SG_2, HG_1, \hat{\xi})$ is a Gelfand triple.

Corollary 4.10.
Let χ be an irreducible character of G. Then

$$\nu_2^{\xi}(\chi) = \frac{1}{|G|}\sum_{g \in G} \chi(g^2)\overline{\xi(g)} = \begin{cases} \pm 1 & (\chi = \xi \otimes \overline{\chi}), \\ 0 & (\chi \neq \xi \otimes \overline{\chi}). \end{cases}$$

Conjugacy class C is a real (resp. complex) $\Leftrightarrow$ $C = C^{-1}$ (resp. $C \neq C^{-1}$).

Corollary 4.11.

$$\frac{1}{2}|\{\chi \mid \nu_2^{\xi}(\chi) = 0\}| = \frac{1}{2}|\{C \mid C \neq C^{-1}\}| + |\{C \mid C = C^{-1}; \xi(C) = -1\}|.$$

Example 4.12. $G = \mathrm{GL}_2(\mathbb{F}_3)$, $\xi = \chi_2$

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	
	R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_7	$\nu_2^{\xi}(\chi)$
χ_1	1	1	1	1	1	1	1	1	0
$\xi = \chi_2$	1	1	1	1	1	-1	-1	-1	0
χ_3	2	2	2	-1	-1	0	0	0	-1
χ_4	3	3	-1	0	0	1	-1	-1	0
χ_5	3	3	-1	0	0	-1	1	1	0
χ_6	2	-2	0	-1	1	0	$\sqrt{2}i$	$-\sqrt{2}i$	-1
χ_7	2	-2	0	-1	1	0	$-\sqrt{2}i$	$\sqrt{2}i$	-1
χ_8	4	-4	0	1	-1	0	0	0	-1

- $\{\chi \mid \nu_2^{\xi}(\chi) = 0\} = \{\chi_1, \chi_2, \chi_4, \chi_5\}$, $(\chi_1 = \xi \otimes \overline{\chi_2}$ and $\chi_4 = \xi \otimes \overline{\chi_5})$.
- $\{C \mid C \neq C^{-1}\} = \{C_7, C_8\}$, $\{C \mid C = C^{-1}; \xi(C) = -1\} = \{C_6\}$

$$1/2 \times 4 = 1/2 \times 2 + 1$$

5 Gelfand Triple $(SG_{2n}, HG_n, \Theta_{\xi,\pi})$

Theorem 5.1.
(1) $\Theta_{\xi,\pi} \uparrow_{HG_n}^{SG_{2n}}$ is multiplicity-free as SG_{2n}-module.
(2) Basis associated with the double cosets can be described.

We explain this theorem, by using an example $\underline{G = \mathrm{GL}_2(\mathbb{F}_3)}$ and $\underline{\xi = \chi_2}$.

Recall
($\star$) Irreducible representaion of SG_{2n} (Put $n_i = |\lambda(\chi_i)|$.):

$$\{S(\lambda(\chi_1),\lambda(\chi_2),\cdots,\lambda(\chi_8)) = \bigotimes_{i=1}^{8}(V_{\chi_i}^{\otimes n_i} \otimes S^{\lambda(\chi_i)})\uparrow_{SG(\underline{n})}^{SG_{2n}} \mid |\underline{\lambda}| = n\}$$

($\star\star$) Fix a complete representatives of $\underline{HG_n \backslash SG_{2n} / HG_n}$ by

$$X = \{x(\underline{\rho}) \mid CT(x(\underline{\rho})) = (\rho(R_1), \rho(R_2), \cdots, \rho(R_7)) \ \& \ |CT(x(\underline{\rho}))| = n\}$$

$\underline{\pi = 1, \delta}$

(1) $\Theta_{\xi,1} \uparrow_{HG_n}^{SG_{2n}} = \bigoplus S(\lambda^1, \lambda^1, (2\lambda^3)', \lambda^4, \lambda^4, (2\lambda^6)', (2\lambda^7)', (2\lambda^8)')$

(2) $\Theta_{\xi,\delta} \uparrow_{HG_n}^{SG_{2n}} = \bigoplus S(\lambda^1, \lambda^1, (2\lambda^3), \lambda^4, \lambda^4, (2\lambda^6), (2\lambda^7), (2\lambda^8))$

- **Index set of irred. comp.**

	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8
	$\chi_1 = \xi \otimes \overline{\chi_2}$		χ_3	$\chi_4 = \xi \otimes \overline{\chi_5}$		χ_6	χ_7	χ_8
$\lambda^i \in$	P		P	P		P	P	P
$\nu_2^\xi(\chi_i)$	0	0	-1	0	0	-1	-1	-1

- $\mathcal{H}^\Theta = \mathrm{Span}\{e_\theta x e_\theta \mid x \in X, CT(x) = (\rho(R_1), \cdots, \rho(R_7)) \in \mathcal{P}_\Theta\}$

	R_1	R_2	R_3	R_4	R_5	R_6	R_7
ξ *on* C_i	1	1	1	1	1	-1	
$R_i =$	C_1	C_2	C_3	C_4	C_5	C_6	$C_7 \cup C_8$
$\mathcal{P}_\Theta =$	P	$\times P$	$\times P$	$\times P$	$\times P$	$\times\{\emptyset\}$	$\times P$

$\pi = \varepsilon, \delta \otimes \varepsilon$

(3) $\Theta_{\xi,\varepsilon} \uparrow_{HG_n}^{SG_{2n}} = \bigoplus S(\lambda^1, \lambda^{1\prime}, D(\lambda^3)', \lambda^4, \lambda^{4\prime}, D(\lambda^6)', D(\lambda^7)', D(\lambda^8)')$

(4) $\Theta_{\xi,\delta\otimes\varepsilon} \uparrow_{HG_n}^{SG_{2n}} = \bigoplus S(\lambda^1, \lambda^{1\prime}, D(\lambda^3), \lambda^4, \lambda^{4\prime}, D(\lambda^6), D(\lambda^7), D(\lambda^8))$

- **Index set of irred. comp.**

	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8
	$\chi_1 = \xi \otimes \overline{\chi_2}$		χ_3	$\chi_4 = \xi \otimes \overline{\chi_5}$		χ_6	χ_7	χ_8
$\lambda^i \in$	P		SP	P		SP	SP	SP
$\nu_2^{\xi}(\chi_i)$	0	0	-1	0	0	-1	-1	-1

- $\mathcal{H}^{\Theta} = \mathrm{Span}\{e_\theta x e_\theta \mid x \in X, CT(x) = (\rho(R_1), \cdots, \rho(R_7)) \in \mathcal{P}_\Theta\}$

	R_1	R_2	R_3	R_4	R_5	R_6	R_7
ξ *on* C_i	1	1	1	1	1	-1	
$R_i =$	C_1	C_2	C_3	C_4	C_5	C_6	$C_7 \cup C_8$
$\mathcal{P}_\Theta =$	OP	$\times OP$	$\times OP$	$\times OP$	$\times OP$	$\times EP$	$\times P$

$\Theta_{\xi,\pi}$-Spherical Function

Roughly speaking, each $\Theta_{\xi,\pi}$-Spherical Functions appear as the coefficients of "multi-component power sum" in some product of symmetric function.

$\star$ $\Theta_{\xi,1}\uparrow_{HG_n}^{SG_{2n}}=\bigoplus S(\lambda^1,\lambda^1,(2\lambda^2)',\lambda^3,\lambda^3,(2\lambda^4)',(2\lambda^5)',(2\lambda^6)')$

$\Theta_{\xi,\pi}$-S.F.$\leftrightarrow$ coeff. of "pow. sum" in $s_{\lambda^1}z_{(\lambda^2)'}s_{\lambda^3}z_{(\lambda^4)'}z_{(\lambda^5)'}z_{(\lambda^6)'}$

$\star$ $\Theta_{\xi,\varepsilon}\uparrow_{HG_n}^{SG_{2n}}=\bigoplus S(\lambda^1,\lambda^{1\prime},D(\lambda^2)',\lambda^3,\lambda^{3\prime},D(\lambda^4)',D(\lambda^5)',D(\lambda^6)')$

$\Theta_{\xi,\pi}$-S.F.$\leftrightarrow$ coeff. of "pow. sum" in $s_{\lambda^1}Q_{\lambda^2}s_{\lambda^3}Q_{\lambda^4}Q_{\lambda^5}Q_{\lambda^6}$.