Physical Combinatorics of Tau Function and Bethe Ansatz

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Tau function of KP hierarchy

$$\tau_i(x) = \langle i | e^{H(x)} \exp\Big(\sum_{j=1}^{N} c_j \psi(p_j)\psi^*(q_j)\Big) | i \rangle$$

$(e^{H(x)} =$ time evolution op. involving $\beta_1, \beta_2, \ldots)$

$$\tau_i(x) = \det(1 + F)$$

$$= 1 + \sum_{1 \le j \le N} F_{jj} + \sum_{1 \le j_1 < j_2 \le N} \begin{vmatrix} F_{j_1 j_1} & F_{j_1 j_2} \\ F_{j_2 j_1} & F_{j_2 j_2} \end{vmatrix} + \cdots ,$$

$$F_{jl} = \frac{c_j q_j}{p_j - q_l} \left(\frac{p_j}{q_j}\right)^{i-1} \prod_m \frac{\beta_m - q_j}{\beta_m - p_j}$$

Ultradiscretization (tropical variable change)

$$\lim_{\epsilon \to +0} \epsilon \log \left(\exp(\frac{a}{\epsilon}) + \exp(\frac{b}{\epsilon}) \right) = \max(a, b)$$

$$\times \qquad\qquad\qquad\qquad a + b$$

keeps the distributive law

$$ab + ac = a(b + c) \longrightarrow \max(a + b, a + c) = a + \max(b, c).$$

Ultradiscrete limit

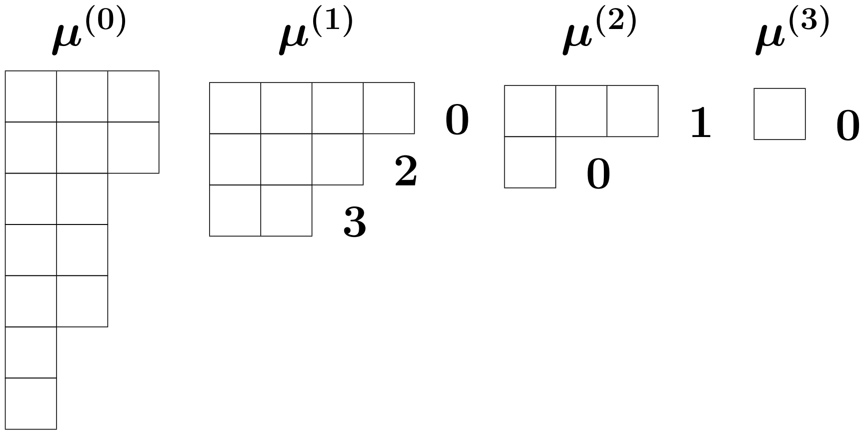
$$\lim_{\epsilon \to +0} \epsilon \log \tau_i(x)$$

with an elaborate ϵ-tuning of parameters

$$c_j, p_j, q_j, \beta_m \quad \text{in} \quad \tau_i(x)$$

leads to an tropical tau function
associated with Rigged Configuration.

Example from $sl_{n=4}$



Rigged configuration

$$(\mu, r) = (\mu^{(0)}, (\mu^{(1)}, r^{(1)}), \ldots, (\mu^{(n-1)}, r^{(n-1)}))$$

$\mu^{(a)}$: configuration (Young diagram)

$r^{(a)}$: rigging (integers attached to $\mu^{(a)}$)

(+ selection rule)

Charge of rigged configuration

$$c(\mu, r) = \frac{1}{2} \sum_{a,b=1}^{n-1} C_{ab} \min(\mu^{(a)}, \mu^{(b)}) - \min(\mu^{(0)}, \mu^{(1)}) + \sum_{a=1}^{n-1} |r^{(a)}|$$

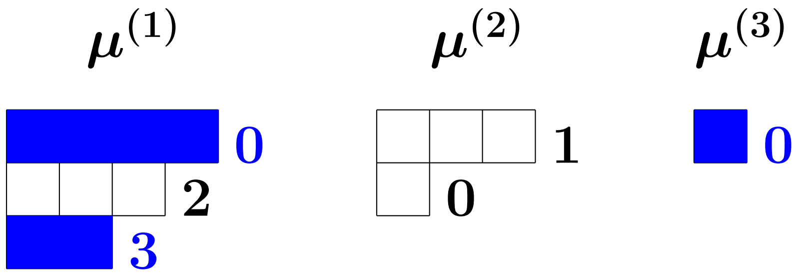
$$\left(\begin{array}{c} \min(\lambda, \mu) = \sum_{ij} \min(\lambda_i, \mu_j), \quad |r| = \sum_i r_i \\ \\ (C_{ab}) = \text{Cartan matrix of } sl_n \end{array} \right)$$

Tropical tau function

$$\boldsymbol{\tau}_i(\boldsymbol{\lambda}) := -\min_{(\boldsymbol{\nu},\boldsymbol{s})}\{c(\boldsymbol{\nu},\boldsymbol{s}) + |\boldsymbol{\nu}^{(i)}|\} \quad (1 \le i \le n)$$

$\min_{(\boldsymbol{\nu},\boldsymbol{s})}$: over $\forall(\boldsymbol{\nu}^{(a)}, \boldsymbol{s}^{(a)}) \subseteq (\boldsymbol{\mu}^{(a)}, \boldsymbol{r}^{(a)})$ such that $\boldsymbol{\nu}^{(0)} = \boldsymbol{\lambda}$.

example :



<u>Proposition</u> (Tropical Hirota equation)

$$\bar{\boldsymbol{\tau}}_{k,i-1} + \boldsymbol{\tau}_{k-1,i} = \max(\bar{\boldsymbol{\tau}}_{k,i} + \boldsymbol{\tau}_{k-1,i-1}, \bar{\boldsymbol{\tau}}_{k-1,i-1} + \boldsymbol{\tau}_{k,i} - \boldsymbol{\mu}_k^{(0)}),$$

where

$$\boldsymbol{\tau}_{k,i} = \boldsymbol{\tau}_i((\boldsymbol{\mu}_1^{(0)}, \ldots, \boldsymbol{\mu}_k^{(0)})), \quad \bar{\boldsymbol{\tau}}_{k,i} = \boldsymbol{\tau}_{k,i}|_{\boldsymbol{r}^{(a)} \to \boldsymbol{r}^{(a)} + \delta_{a1}\boldsymbol{\mu}^{(1)}}$$

Rigged configuration originates in string hypothesis in Bethe ansatz

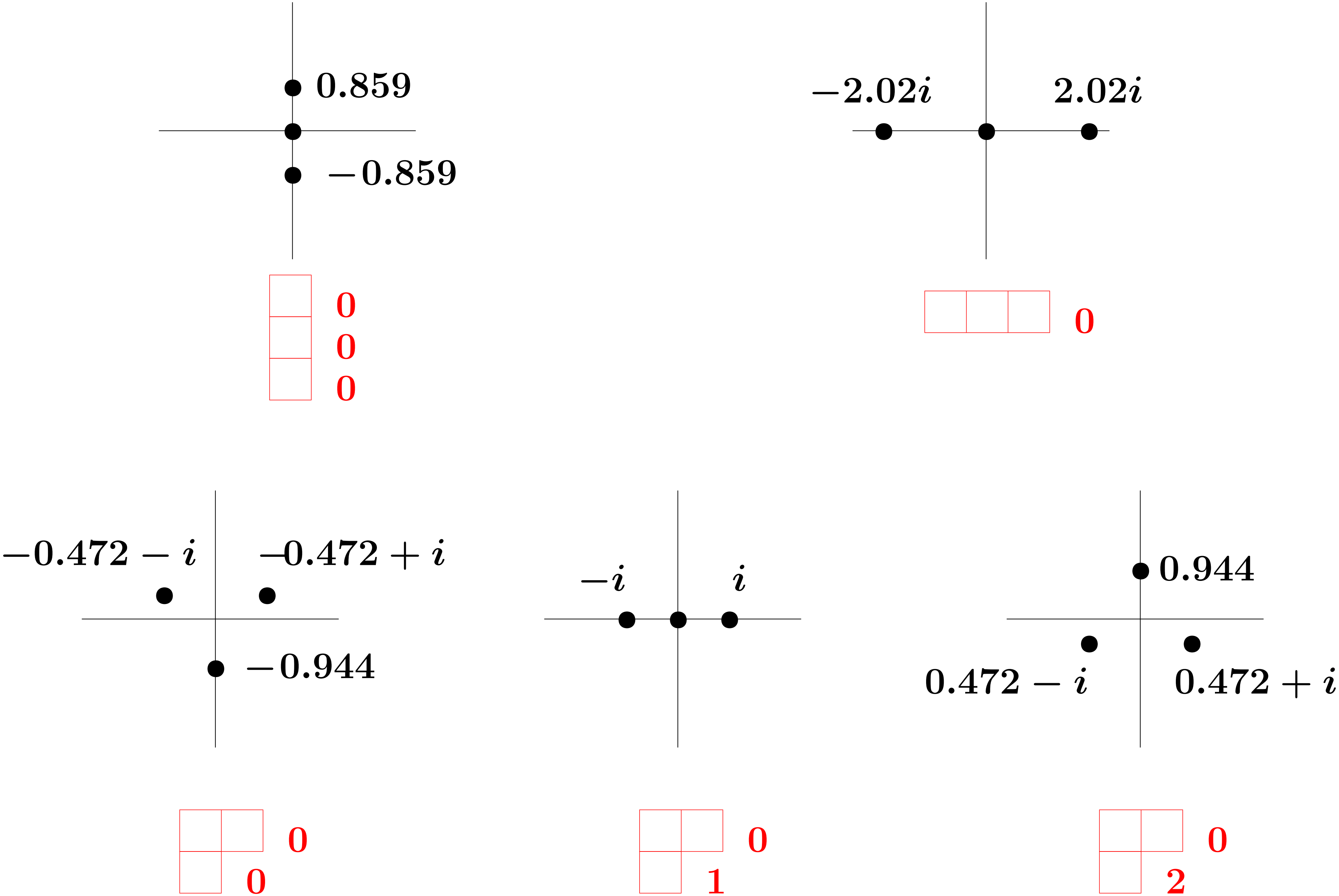
Solitons in tau function = Strings in Bethe ansatz

Example from sl_2 (Heisenberg chain)

$$H = \sum_{k=1}^{L} (\sigma_k^x \sigma_{k+1}^x + \sigma_k^y \sigma_{k+1}^y + \sigma_k^z \sigma_{k+1}^z) \quad : \quad (\mathbb{C}^2)^{\otimes L} \longrightarrow (\mathbb{C}^2)^{\otimes L}$$

Bethe equation for length $L = 6$ chain with 3 down spins.

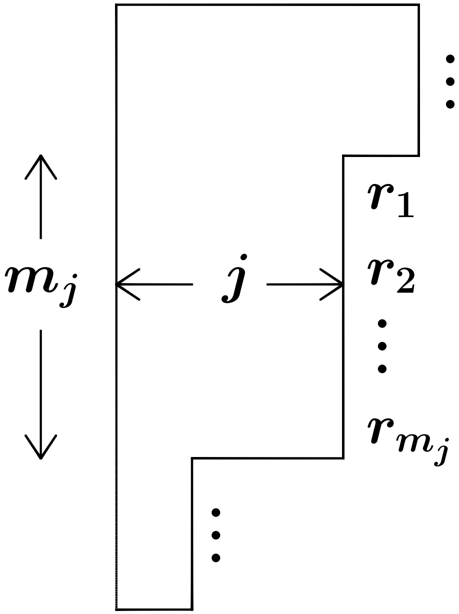
$$\left(\frac{u_1 + i}{u_1 - i}\right)^6 = \frac{(u_1 - u_2 + 2i)(u_1 - u_3 + 2i)}{(u_1 - u_2 - 2i)(u_1 - u_3 - 2i)},$$
$$\left(\frac{u_2 + i}{u_2 - i}\right)^6 = \frac{(u_2 - u_1 + 2i)(u_2 - u_3 + 2i)}{(u_2 - u_1 - 2i)(u_2 - u_3 - 2i)},$$
$$\left(\frac{u_3 + i}{u_3 - i}\right)^6 = \frac{(u_3 - u_1 + 2i)(u_3 - u_2 + 2i)}{(u_3 - u_1 - 2i)(u_3 - u_2 - 2i)}.$$



0.859
−0.859
0
0
0
−2.02i
2.02i
0
−0.472 − i
−0.472 + i
−0.944
0
0
−i
i
0
1
0.944
0.472 − i
0.472 + i
0
2

Rigged configuration for sl_2 (spin $\frac{1}{2})^{\otimes L}$

Young diagram = configuration, $\{r_i\}$ = rigging



$$0 \le r_1 \le \cdots \le r_{m_j} \le p_j$$

$\cdots$ (fermionic) selection rule

$$p_j = L - 2\sum_{k\ge 1} \min(j,k) m_k$$

$\cdots$ vacancy number

$$\sharp \text{ of rigged configurations} = \sum_{\{m_i\}} \prod_{i\ge 1} \binom{p_i + m_i}{m_i}$$

Bethe's fermionic formula (1931)

$$\text{Kostka number}\quad K_{(L-N,N),(1^L)} = \sum_{\{m_i\}} \prod_{i \geq 1} \binom{p_i + m_i}{m_i}$$

{ Bethe roots } ↔ (1 : 1?) { irreducible sl_2 components }

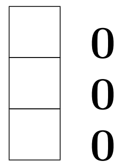
↕ 1 : 1 rigorously

{ Bethe roots } ↔ (1 : 1?) { rigged configuration }

KKR theory. (Kerov-Kirillov-Reshetikhin 1986)

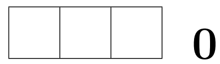
Invention of rigged configuration, canonical bijection and q-analogue of Bethe's formula from integrable spin chain with sl_n symmetry.

$$\{\text{rigged configurations}\} \overset{\color{blue}{\text{KKR}}}{\longleftrightarrow} \{\text{standard tableaux}\} \overset{\color{red}{\text{RS}}}{\longleftrightarrow} \{\text{highest paths}\}$$



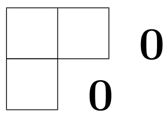
$$\begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4&6\\\hline\end{array}$$

121212



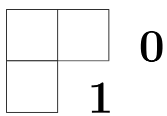
$$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5&6\\\hline\end{array}$$

111222



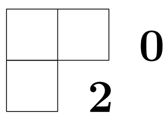
$$\begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5&6\\\hline\end{array}$$

121122



$$\begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4&6\\\hline\end{array}$$

112212



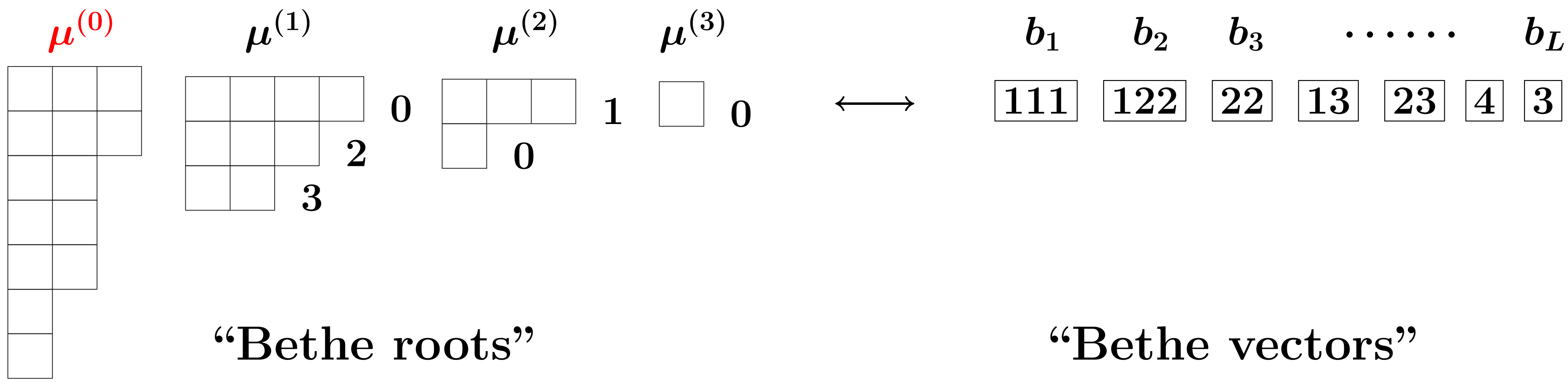
$$\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5&6\\\hline\end{array}$$

112122

"composition of our bijection with the Robinson-Schensted-Knuth correspondence may be viewed as a combinatorial version of the Bethe ansatz."([KKR 1986])

KKR bijection for sl_n

$$\{\text{rigged configurations}\} \overset{1:1}{\longleftrightarrow} \{\text{highest paths}\}$$

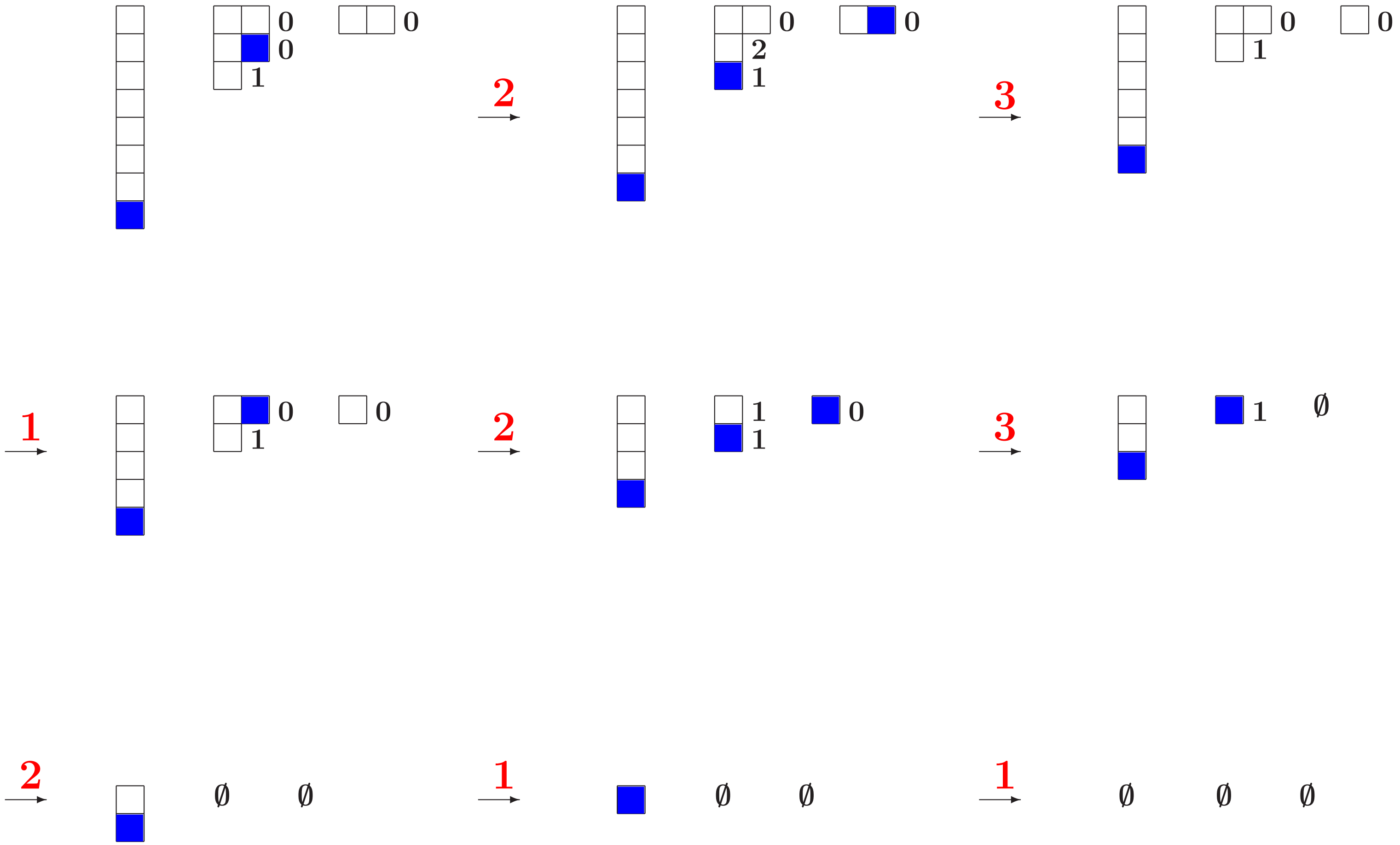


highest path $= b_1\, b_2\, \ldots\, b_L$

b_i = row shape $(\mu_i^{(0)})$ semistandard tableau.

(+ highest condition)

Example of KKR algorithm from sl_3



Top left rigged configuration $\xrightarrow{\text{KKR}}$ **11232132**

Theorem.(K-Sakamoto-Yamada 2007)

Image of the KKR map

$$(\mu^{(0)}, (\mu^{(1)}, r^{(1)}), \ldots, (\mu^{(n-1)}, r^{(n-1)})) \overset{\text{KKR}}{\longmapsto} b_1 \ldots b_L$$

$$b_k = (\overbrace{1 \ldots 1}^{x_{k,1}}, \ldots, \overbrace{n \ldots n}^{x_{k,n}}) \quad \text{(semistandard tableau)},$$

is given by

$$\boxed{x_{k,i} = \tau_{k,i} - \tau_{k-1,i} - \tau_{k,i-1} + \tau_{k-1,i-1}}$$

We will see that this is an analogue of

$$u = 2\frac{\partial^2 \log \tau}{\partial x^2}$$

for KdV eq. in tropical (ultradiscrete) soliton theory.

Crystals and combinatorial R for $U_q(\widehat{sl}_n)$

$$B_l = \{ \boxed{i_1, \ldots, i_l} \mid \text{semistandard}\}$$

$$\mathrm{Aff}(B_l) = \{ \boxed{i_1, \ldots, i_l}_{\,d} \in B_l \times \mathbb{Z}\}$$

equipped with crystal structures.

Example:

$\boxed{1233} \otimes \boxed{124} \in B_4 \otimes B_3, \qquad \boxed{1233}_5 \otimes \boxed{124}_9 \in \mathrm{Aff}(B_4) \otimes \mathrm{Aff}(B_3).$

$u_l := \boxed{11\ldots1} \in B_l$ is the (classically) highest element.

A path is an element $b_1 \otimes b_2 \otimes \cdots \in B_{l_1} \otimes B_{l_2} \otimes \cdots.$

Combinatorial R (classical part)

$$R \ : \ \ B_l \otimes B_m \xrightarrow{\sim} B_m \otimes B_l, \qquad x \otimes y \mapsto \tilde{y} \otimes \tilde{x}$$

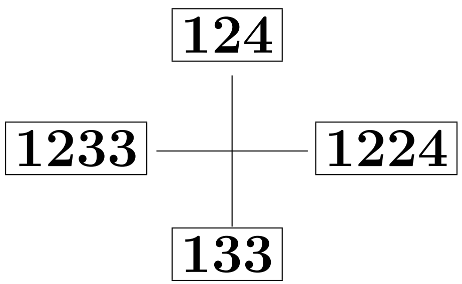
$$\tilde{x}_i - x_i = y_i - \tilde{y}_i = Q_i(x \otimes y) - Q_{i-1}(x \otimes y) \quad (i \bmod n),$$

$$x_i = \sharp \text{ of letter } i \text{ in tableau } x \quad (y_i : \text{similar}),$$

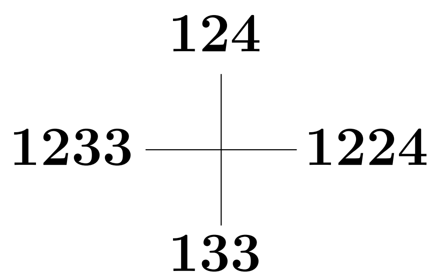
$$Q_i(x \otimes y) = \min_{1 \le k \le n} \{ \sum_{j=1}^{k-1} x_{i+j} + \sum_{j=k+1}^{n} y_{i+j} \} \quad \cdots \quad i \text{ th local energy.}$$

Example : $\boxed{1233} \otimes \boxed{124} \simeq \boxed{133} \otimes \boxed{1224}$

will be denoted by

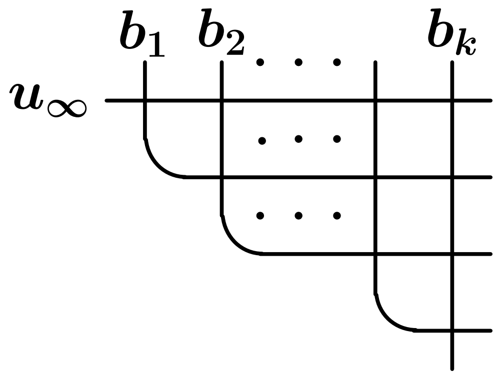


or simply



Energy $\mathcal{E}_i$ of path $b_1 \otimes \cdots \otimes b_k$ (i mod n)

$\mathcal{E}_i(b_1 \otimes \cdots \otimes b_k) :=$ **Sum of $Q_i(x \otimes y)$ attached to all vertices in**



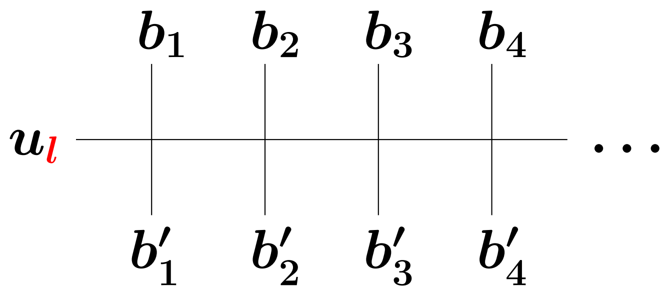
Theorem.([KSY] "tropical fermionic formula")

Suppose $b_1 \otimes \cdots \otimes b_L \overset{\text{KKR}}{\longleftrightarrow} (\mu, r) \longrightarrow \{\tau_{k,i}\}$. **Then,**

$$\mathcal{E}_i(b_1 \otimes \cdots \otimes b_k) = \tau_{k,i} \quad (1 \leq k \leq L) \qquad \blacksquare$$

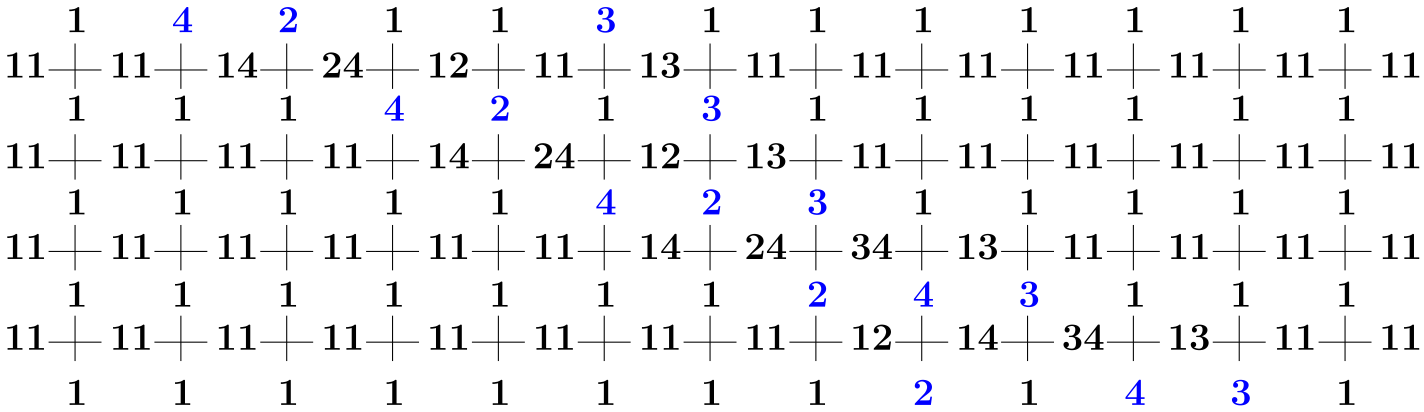
Fusion $U_q(\widehat{sl}_n)$ vertex model at $q = 0$

$$
\begin{array}{rccc}
T_l : & B_1 \otimes B_1 \otimes B_1 \otimes \cdots & \longrightarrow & B_1 \otimes B_1 \otimes B_1 \otimes \cdots \\
 & b_1 \otimes b_2 \otimes b_3 \otimes \cdots & \longmapsto & b'_1 \otimes b'_2 \otimes b'_3 \otimes \cdots
\end{array}
$$



$T_1, T_2, \ldots$: commuting family of time evolutions
(deterministic fusion transfer matrices)

Example of time evolution T_2 :



The dynamics on vertical edges reproduces Box-ball system (Takahashi-Satsuma 1990).

$$\cdots 1\,42\,11\,3\,1111111 \cdots$$
$$\cdots 111\,42\,1\,3\,111111 \cdots$$
$$\cdots 11111\,423\,11111 \cdots$$
$$\cdots 1111111\,243\,111 \cdots$$
$$\cdots 11111111\,2\,1\,43\,1 \cdots$$

1= empty box, $\mathbf{2, 3, 4}$ = colored balls.

Soliton = consecutive array of balls $i_1 \ldots i_s$ with color $i_1 \geq \cdots \geq i_s$

Collision of 3 solitons

$$\cdots 11432114211113111111111111111 \cdots$$
$$\cdots 11111432142111311111111111111 \cdots$$
$$\cdots 11111111431422131111111111111 \cdots$$
$$\cdots 11111111114311423211111111111 \cdots$$
$$\cdots 11111111111143112143211111111 \cdots$$
$$\cdots 11111111111111431211143211111 \cdots$$
$$\cdots 11111111111111114132111143211 \cdots$$

$$\cdots 11432111142113111111111111111 \cdots$$
$$\cdots 11111432111421311111111111111 \cdots$$
$$\cdots 11111111432114231111111111111 \cdots$$
$$\cdots 11111111111432124311111111111 \cdots$$
$$\cdots 11111111111111413243211111111 \cdots$$
$$\cdots 11111111111111141132143211111 \cdots$$
$$\cdots 11111111111111114111321143211 \cdots$$

Yang-Baxter relation is valid.

(Solitons in final state are independent of the order of collisions.)

Theorem.([KSY])

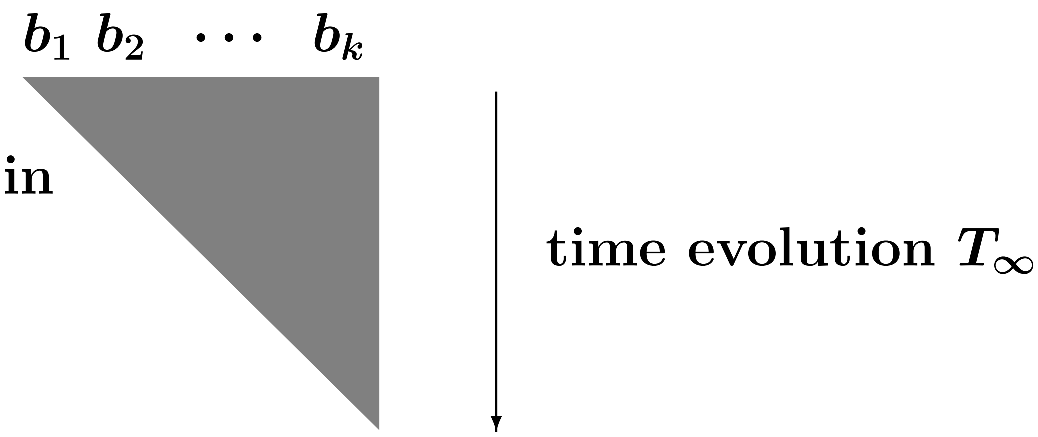
(1) **Tropical tau function**

$=$ **Energy of affine crystal** (math) $\cdots$ previous theorem

$=$ Baxter's **corner transfer matrix** for box-ball system (phys)

Let $b_1 b_2 \cdots b_L \overset{\text{KKR}}{\longleftrightarrow} (\mu, r) \longrightarrow \{\tau_{k,i}\}$. Then,

$\tau_{k,i} = \sharp$ of "balls" in

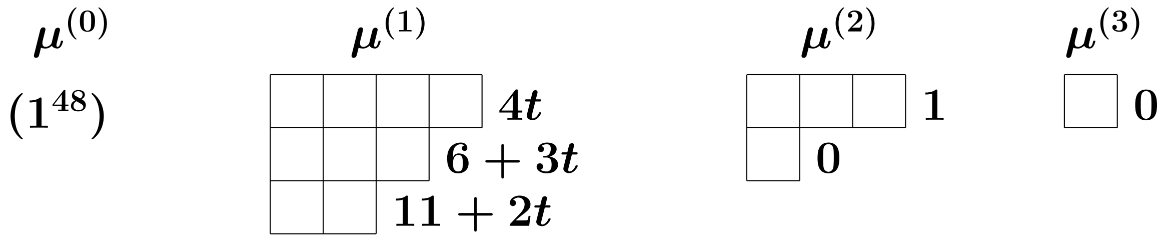


(2) Tropical Hirota equation = eq. of motion of box-ball system.

	Bethe ansatz	Corner transfer matrix
main combinatorial object	rigged configuration	energy (charge) in affine crystal
role in box-ball system	action-angle variable	tau function
dynamics	linear	bilinear

Dynamics of box-ball system in terms of rigged configuration

$t=0$: 111122221111133211431111111111111111111111111111
$t=1$: 111111112222111133214311111111111111111111111111
$t=2$: 111111111111222211332431111111111111111111111111
$t=3$: 111111111111111122221132433111111111111111111111
$t=4$: 111111111111111111112221322433111111111111111111
$t=5$: 111111111111111111111112211322433211111111111111
$t=6$: 111111111111111111111111122111322143321111111111
$t=7$: 111111111111111111111111111221111322114332111111



configuration ··· conserved quantity (action variable)

rigging ··· linear flow (angle variable)

KKR bijection ··· direct/inverse scattering map (separation of variables)

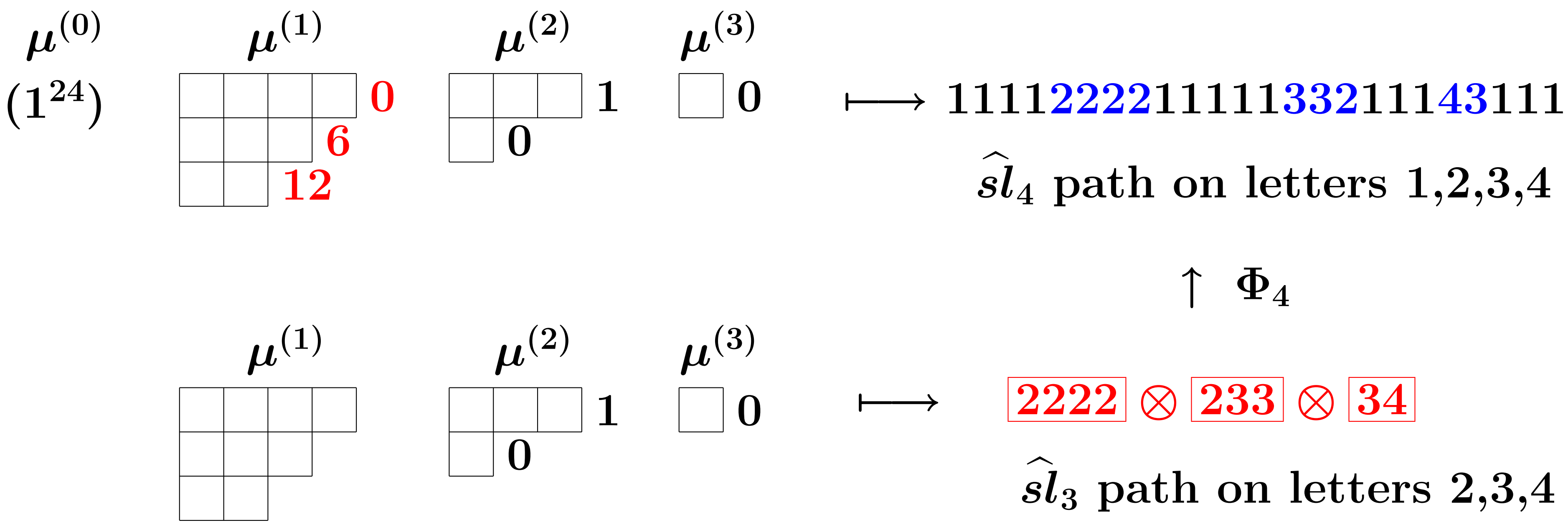
Theorem. ([K-Okado-S-Takagi-Y, S])

- KKR bijection = direct/inverse scattering map of the box-ball system.

- KKR map $= \Phi_n \circ \cdots \circ \Phi_2$

  $\Phi_a =$ composition of $\widehat{sl}_a$ comb. R ("vertex operator")

KKR theory	box-ball system	crystal theory
rigged configuration	scattering data	$\otimes_k \mathrm{Aff}(B_{l_k})$
KKR bijection	direct/inverse scattering	vertex operator



Data colored red specifies the scattering data

$$\boxed{2222}_4 \otimes \boxed{233}_{10} \otimes \boxed{34}_{15} \in \mathrm{Aff}(B_4) \otimes \mathrm{Aff}(B_3) \otimes \mathrm{Aff}(B_2).$$

This procedure applied recursively with respect to rank leads to

$$\text{KKR map} = \Phi_n \circ \Phi_{n-1} \circ \cdots \circ \Phi_2.$$

RHS is a combinatorial version of nested Bethe ansatz (Schultz '83).

Summary so far

$$\tau := -\min_{\text{power set of rigged conf.}} \{\text{charge}\}$$

- τ = tropical analogue of KP tau (satisfies tropical Hirota eq.)

  = affine crystal energy

  = "corner transfer matrix" of box-ball system

- KKR map $= \tau - \tau - \tau + \tau$

  $= \Phi_n \circ \cdots \circ \Phi_2$

KKR theory = inverse scattering scheme of box-ball system
on ∞ lattice

An example of generalizations.

- integrable cellular automata associated with affine Lie algebras

$D_5^{(1)}$ -automaton

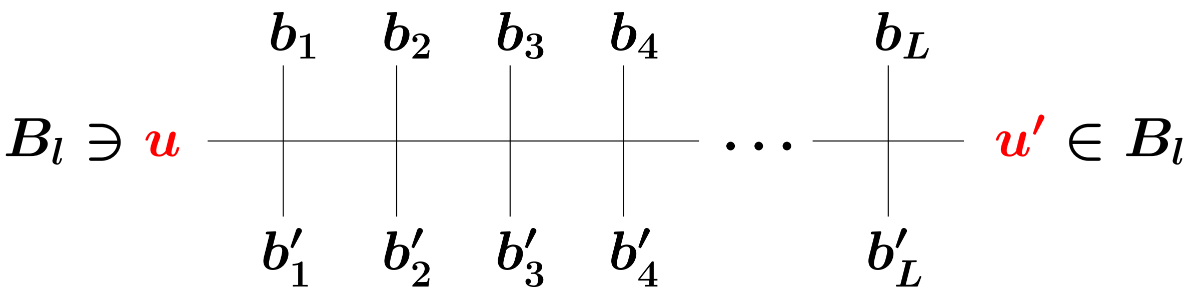
$$
\begin{array}{l}
\cdots 111\bar{3}\bar{5}3221111\bar{2}\bar{2}\bar{5}11111111111111111111111111111111 \cdots \\
\cdots 11111111\bar{3}\bar{5}32211\bar{2}\bar{2}\bar{5}11111111111111111111111111111 \cdots \\
\cdots 1111111111111\bar{3}\bar{5}322\bar{2}\bar{2}\bar{5}11111111111111111111111111 \cdots \\
\cdots 111111111111111111\bar{3}\bar{5}3\bar{1}\bar{1}\bar{5}11111111111111111111111 \cdots \\
\cdots 111111111111111111111114\bar{1}\bar{1}\bar{4}\bar{5}\bar{5}111111111111111111 \cdots \\
\cdots 11111111111111111111111111422\bar{2}\bar{2}\bar{4}\bar{5}\bar{5}1111111111111 \cdots \\
\cdots 1111111111111111111111111111142211\bar{2}\bar{2}\bar{4}\bar{5}\bar{5}11111111 \cdots \\
\cdots 111111111111111111111111111111114221111\bar{2}\bar{2}\bar{4}\bar{5}\bar{5}111 \cdots
\end{array}
$$

- particles and anti-particles undergo pair-creations/annihilations

- solitons and their scattering rules characterized by crystal theory. ([HKOTY 2002])

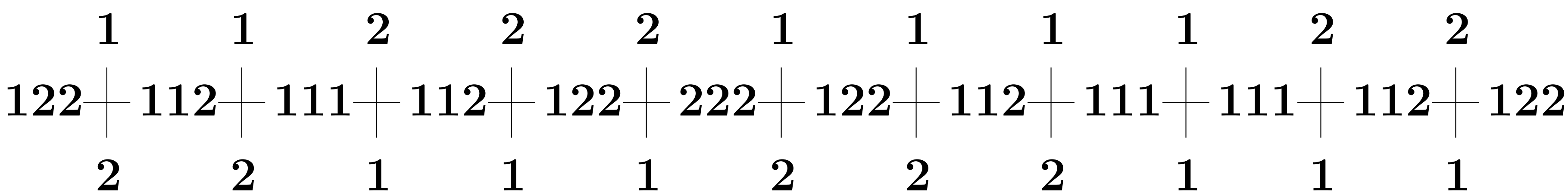
Periodic generalization of KKR theory (sl_2 case)

$$T_l: \; B_1 \otimes B_1 \otimes \cdots \otimes B_1 \longrightarrow B_1 \otimes B_1 \otimes \cdots \otimes B_1$$
$$b_1 \otimes b_2 \otimes \cdots \otimes b_L \longmapsto b'_1 \otimes b'_2 \otimes \cdots \otimes b'_L$$



Choice s.t. $u = u'$ defines periodic box-ball system (Yura et al. 2002)

Example of T_3 : ($B_1 = \{ \boxed{1} \, , \, \boxed{2} \}$)



$T_1, T_2, \ldots$ commuting family of time evolutions.

evolution under T_2

1 1 2 1 1 1 2 2 2 1 1 1 2 2

2 2 1 2 1 1 1 1 2 2 2 1 1 1

1 1 2 1 2 2 1 1 1 1 2 2 2 1

2 1 1 2 1 1 2 2 1 1 1 1 2 2

2 2 2 1 2 1 1 1 2 2 1 1 1 1

1 1 2 2 1 2 2 1 1 1 2 2 1 1

1 1 1 1 2 1 2 2 2 1 1 1 2 2

2 2 1 1 1 2 1 1 2 2 2 1 1 1

evolution under T_3

1 1 2 1 1 1 2 2 2 1 1 1 2 2

2 2 1 2 1 1 1 1 1 2 2 2 1 1

1 1 2 1 2 2 2 1 1 1 1 1 2 2

2 2 1 2 1 1 1 2 2 2 1 1 1 1

1 1 2 1 2 2 1 1 1 1 2 2 2 1

2 2 1 2 1 1 2 2 1 1 1 1 1 2

1 1 2 1 2 2 1 1 2 2 2 1 1 1

1 1 1 2 1 1 2 2 1 1 1 2 2 2

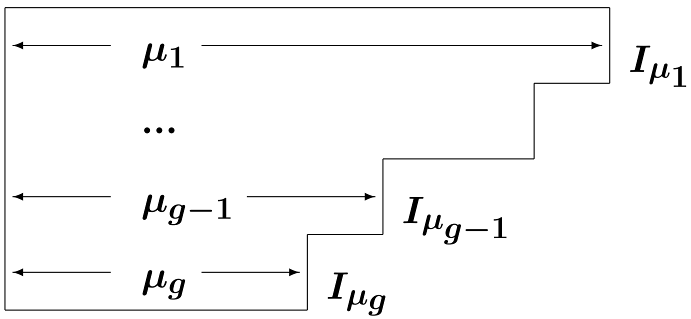
A guide to a decent generalization of KKR theory

Construct an inverse scattering scheme for periodic box-ball system.

• Action-angle variables

any path $\qquad$ highest path $\qquad$ rigged conf.

$$b_1 \ldots\ldots b_L \overset{\text{cyclic shift}}{\longmapsto} b_{d+1} \ldots b_L b_1 \ldots b_d \overset{\text{KKR}}{\longmapsto} (\mu, I) \qquad (d \text{ is not unique})$$



$$\mu = (\mu_1, \ldots, \mu_g), \quad \mathrm{I} = (I_{\mu_1}, \ldots, I_{\mu_g}).$$

For simplicity we assume $\mu_1 > \mu_2 > \cdots > \mu_g$.

Set $p_i := L - 2\sum_{j \in \mu} \min(i, j)$ $\quad$ (vacancy number)

$$\overset{\text{any path}}{b_1 \dots b_L} \overset{\text{cyc.}}{\longmapsto} \overset{\text{highest path}}{b_{d+1} \dots b_L b_1 \dots b_d} \overset{\text{KKR}}{\longmapsto} (\mu, I) \quad (d \text{ is not unique})$$

Lemma.

- μ is independent of d and invariant under $\{T_l\}$ (action variable)
- $(\mathrm{I} + d\,\mathrm{h}_1)/A\mathbb{Z}^g$ is independent of d (angle variable), where

$$\mathrm{h}_l = (\min(l, i))_{i \in \mu} \in \mathbb{Z}^g, \quad A = \big(\delta_{ij} p_i + 2 \min(i, j)\big)_{i,j \in \mu}.$$

Define

$\mathcal{P}(\mu) :=$ {paths whose action variable $= \mu$} iso-level set

$\mathcal{J}(\mu) := \mathbb{Z}^g / A\mathbb{Z}^g$ set of angle variables

$\Phi : \mathcal{P}(\mu) \longrightarrow \mathcal{J}(\mu)$ by $\Phi(b_1 \dots b_L) := (\mathrm{I} + d\,\mathrm{h}_1)/A\mathbb{Z}^g$

Theorem. ([KT-Takenouchi 2006] "Tropical Abel-Jacobi" map)

$\Phi : \mathcal{P}(\mu) \to \mathcal{J}(\mu)$ is a bijection.

$$\begin{array}{ccc} \mathcal{P}(\mu) & \xrightarrow{\Phi} & \mathcal{J}(\mu) \\ T_l \downarrow & & \downarrow T_l \\ \mathcal{P}(\mu) & \xrightarrow{\Phi} & \mathcal{J}(\mu) \end{array}$$

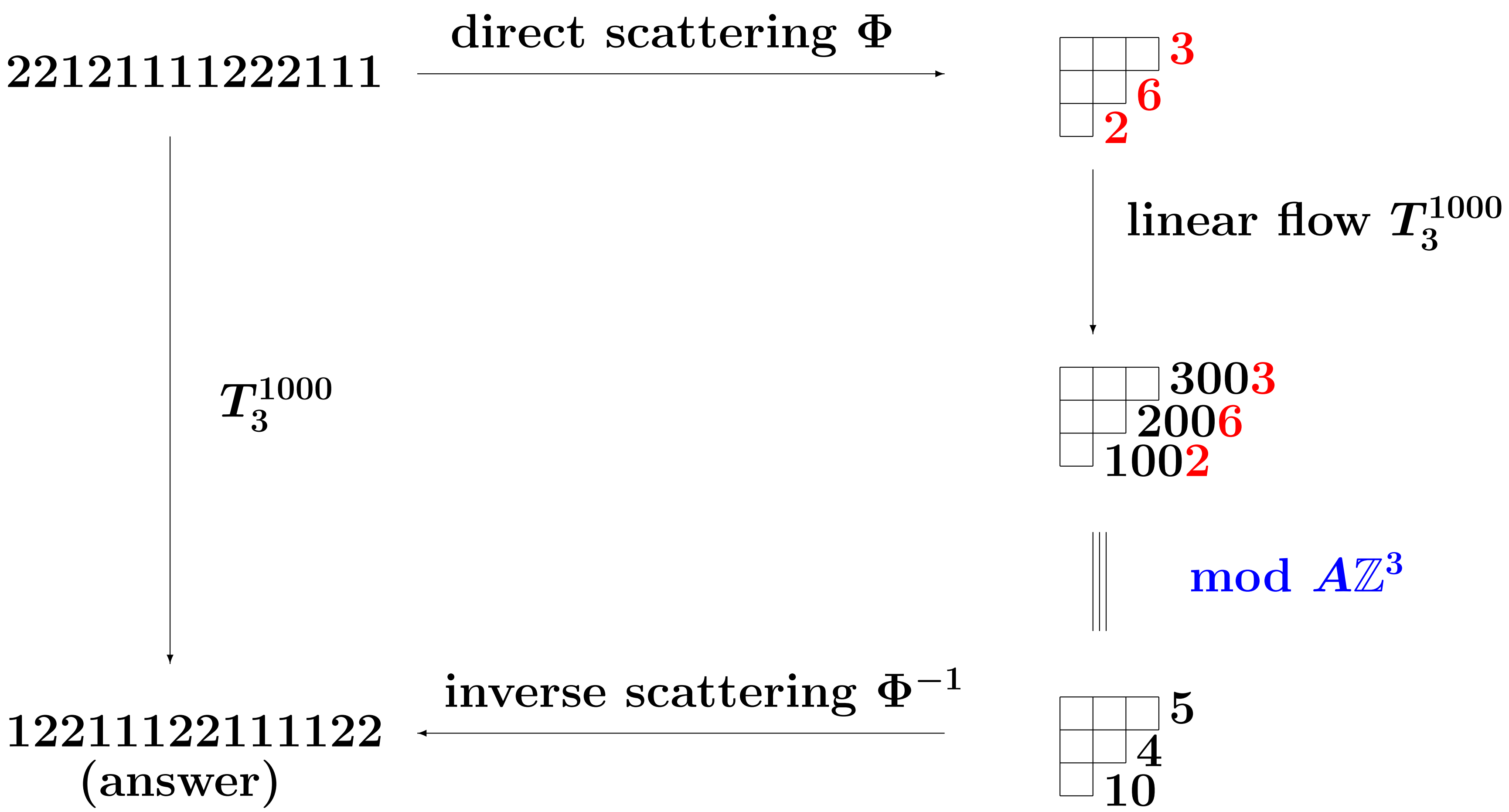
is commutative, where $T_l(\mathrm{J}) = \mathrm{J} + \mathrm{h}_l$ on $\mathcal{J}(\mu)$ ■

Nonlinear dynamics becomes straight motion in

$$\mathcal{J}(\mu) = \mathbb{Z}^g / A\mathbb{Z}^g,$$

which is an tropical analogue of Jacobi variety.

Solution of initial value problem (inverse method)



Riemann theta (with pure imaginary period matrix) :

$$\vartheta(\mathrm{z}) := \sum_{\mathrm{n}\in\mathbb{Z}^g} \exp\Bigl(-\frac{{}^t\mathrm{n}A\mathrm{n}/2 + {}^t\mathrm{n}\mathrm{z}}{\epsilon}\Bigr)$$

Tropical Riemann theta ($\mathrm{z} \in \mathbb{R}^g$):

$$\Theta(\mathrm{z}) := \lim_{\epsilon\to+0} \epsilon \log \vartheta(\mathrm{z}) = -\min_{\mathrm{n}\in\mathbb{Z}^g}\{{}^t\mathrm{n}A\mathrm{n}/2 + {}^t\mathrm{n}\mathrm{z}\}$$

Theorem. ([KS 2006] "Tropical Jacobi inversion")

$$\begin{array}{ccc} \mathcal{J}(\mu) & \to & \mathcal{P}(\mu) \\ (\mu, \mathrm{I}) & \mapsto & b_1\, b_2\, \ldots\, b_L \ \ (\in \{1,2\}^L) \end{array}$$

is given by

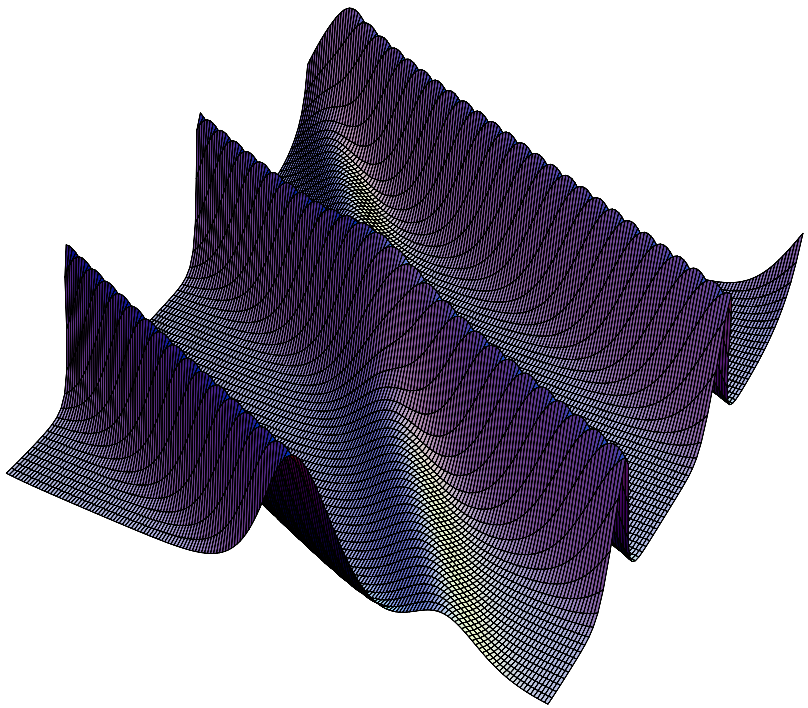
$$\begin{aligned} b_k = 1 &+ \Theta\bigl(\mathrm{J} - k\mathrm{h}_1\bigr) - \Theta\bigl(\mathrm{J} - (k-1)\mathrm{h}_1\bigr) \\ &- \Theta\bigl(\mathrm{J} - k\mathrm{h}_1 + \mathrm{h}_\infty\bigr) + \Theta\bigl(\mathrm{J} - (k-1)\mathrm{h}_1 + \mathrm{h}_\infty\bigr), \end{aligned}$$

with $\mathrm{J} = \mathrm{I} +$ (known constant vector).

Inverse tropicalization: double difference of $\Theta \longrightarrow$ double ratio of ϑ

$$b(k,t) = \frac{\vartheta\big(\mathrm{J} + t\mathrm{h}_\infty - k\mathrm{h}_1\big)\vartheta\big(\mathrm{J} + (t+1)\mathrm{h}_\infty - (k-1)\mathrm{h}_1\big)}{\vartheta\big(\mathrm{J} + t\mathrm{h}_\infty - (k-1)\mathrm{h}_1\big)\vartheta\big(\mathrm{J} + (t+1)\mathrm{h}_\infty - k\mathrm{h}_1\big)}.$$

Same structure as the quasi-periodic solution of the KdV/Toda eq. by Date-Tanaka and Kac-Moerbeke (1976).



Two soliton state with amplitudes 6 and 2.
System size $L = 170$, duration $0 \le t \le 70$.

Origin of tropical period matrix A

$U_q(\widehat{sl}_2)$ Bethe equation at $q = 0$ (string center eq.):

$$A\mathrm{x} \equiv \text{constant vector} \quad \mathrm{mod}\ A\mathbb{Z}^g$$

(K-Nakanishi 2000)

Remark.

$$\text{Bethe root } \mathrm{x} \overset{1:1}{\longleftrightarrow} J \in \mathcal{J}(\mu) = \mathbb{Z}^g / A\mathbb{Z}^g \quad \text{via} \quad A\mathrm{x} = J.$$

$|\mathcal{J}(\mu)|$ = fermionic formula for weight multiplicities.

Combinatorial Bethe ansätze

	$q=1$	$q=0$
fermionic formula	multiplicity of irreps.	weight multiplicity
box-ball system	∞ lattice	periodic lattice
action-angle variable (Bethe roots)	rigged configuration	Sol. of string center eq.

Office configuration of GGKM (Princeton ∼1966)