Dyck partitions, quasi-minuscule quotiens and Kazhdan-Lusztig polynomials

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Nagoya 名古屋 5/9/2008

partly based on a joint work with

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Outline

1. Background
2. Parabolic Kazhdan-Lusztig polynomials
3. Combinatorics of quasi-minuscule quotients
4. $\bullet$-Dyck partitions
5. Main result
6. Open problems
7. Recent enumerative results

1. Background

1.1 Coxeter groups

W : Coxeter group $\quad$ S : set of generators

Set of *reflections* of W: $\quad T = \{vsv^{-1} : v \in W,\ s \in S\}$.

Let $v \in W$. The *length* of v is

$$\ell(v) = \min\{k : v \text{ is a product of } k \text{ generators}\}.$$

The (*right*) *descent set* of v is

$$D(v) = \{s \in S : \ell(vs) < \ell(v)\}.$$

Bruhat graph of W: directed graph with W as vertex set and

$$u \to v \quad \Leftrightarrow \quad u^{-1}v \in T \quad \text{and} \quad \ell(u) < \ell(v).$$

Bruhat order of W: partial order on W defined by

$$u \leqslant v \quad \Leftrightarrow \quad u = u_0 \to u_1 \to \cdots \to u_k = v.$$

W, with the Bruhat order, is a graded poset with rank function ℓ.

For $u, v \in W$, with $u < v$, we set

$$\ell(u,v) = \ell(v) - \ell(u) \quad \text{(distance in the Bruhat order).}$$

Let $J \subseteq S$ be a fixed subset of generators.

The *parabolic subgroup* of W generated by J is

$$W_J = \langle J \rangle.$$

The *quotient* of W by J is

$$W^J = \{v \in W : \ell(sv) > \ell(v) \text{ for all } s \in J\}.$$

We will consider particular quotients of the symmetric group.

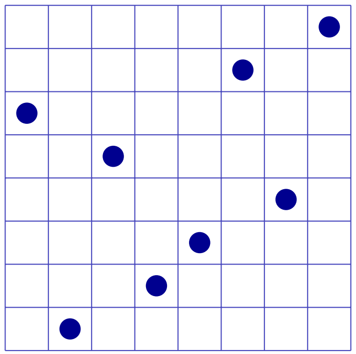
1.2 The symmetric group

$\mathbf{P} = \{1, 2, 3, \ldots\}$, $\quad [n] = \{1, 2, \ldots, n\} \quad (n \in \mathbf{P})$,

Symmetric group: $\quad S_n = \{v : [n] \to [n] \text{ bijection}\}$.

We denote $v \in S_n$ by the word $v(1)\, v(2) \ldots v(n)$ and by its *diagram*.

Example. $v = 61523748 \in S_8$ has diagram



S_n is a Coxeter group, with generators the simple transpositions:

$$S = \{(1,2),(2,3),\ldots,(n-1,n)\}.$$

When we refer to these generators, the transposition $(i, i+1)$ is simply denoted by i. With this convention, the set of generators of S_n is

$$S = [n-1].$$

The reflections are all the transpositions:

$$T = \{(i,j) \in [n]^2 : i < j\}.$$

Let $v \in S_n$. The length of v is the number of its inversions:

$$\ell(v) = |\{(i,j) \in [n]^2 : i < j \text{ and } v(i) > v(j)\}|.$$

The descent set of v is

$$D(v) = |\{i \in [n-1] : v(i) > v(i+1)\}|.$$

Let $J \subseteq [n-1]$. The quotient of S_n by J is

$$(S_n)^J = \{v \in S_n : v^{-1}(r) < v^{-1}(r+1) \text{ for all } r \in J\}.$$

The *maximal quotients* of S_n are obtained by taking

$$J = [n-1] \setminus \{i\} \quad (i \in [n-1]).$$

The *quasi-minuscule quotients* of S_n are obtained by taking

$$J = [n-1] \setminus \{i-1, i\} \quad (2 \leqslant i \leqslant n-1)$$

or

$$J = [n-1] \setminus \{1, n-1\}.$$

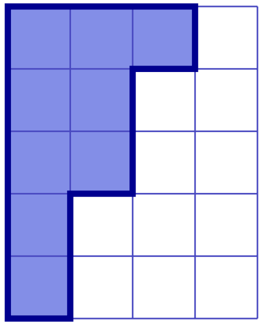
In this talk we study the quasi-minuscule quotiens of S_n.

1.3 Partitions and lattice paths

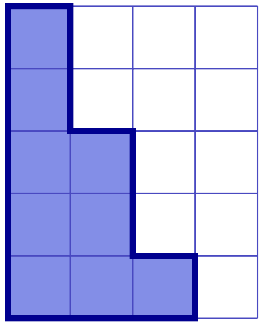
We identify a partition $\lambda = (\lambda_1, \ldots, \lambda_k) \subseteq (n^m)$ with its *diagram*:

$$\{(i,j) \in \mathbf{P}^2 : 1 \leqslant i \leqslant k \text{ and } 1 \leqslant j \leqslant \lambda_i\}.$$

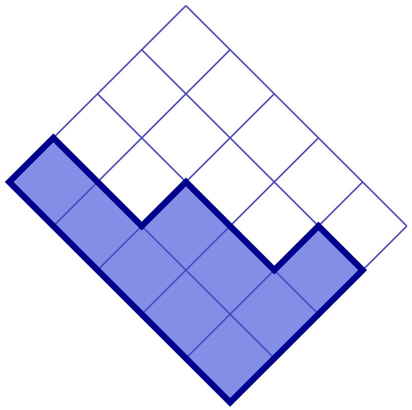
Example. $\lambda = (3,2,2,1,1) \subseteq (4^5)$.



English notation



French notation



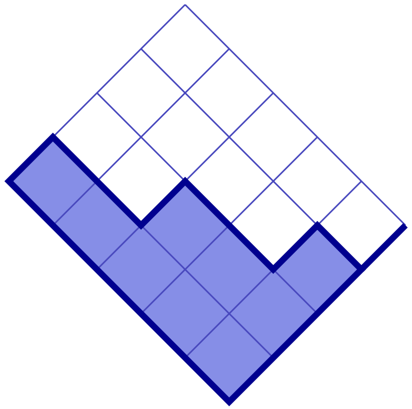
Our notation (Japanese?)

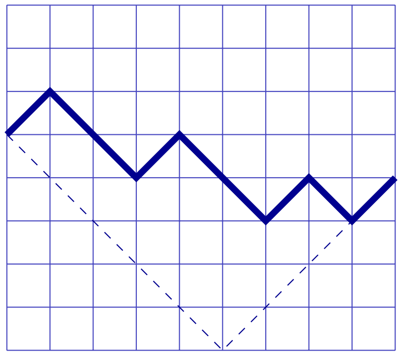
Given a partition $\lambda \subseteq (n^m)$, the *path* associated with λ is the lattice path from $(0, m)$ to $(n+m, n)$, with steps $(1,1)$ (up steps) and $(1,-1)$ (down steps) which is the upper border of the diagram of λ:

$$\text{path}(\lambda) = x_1 x_2 \dots x_{n+m}, \quad \text{with } x_k \in \{\mathsf{U}, \mathsf{D}\},$$

Note that path(λ) has exactly n U's and m D's.

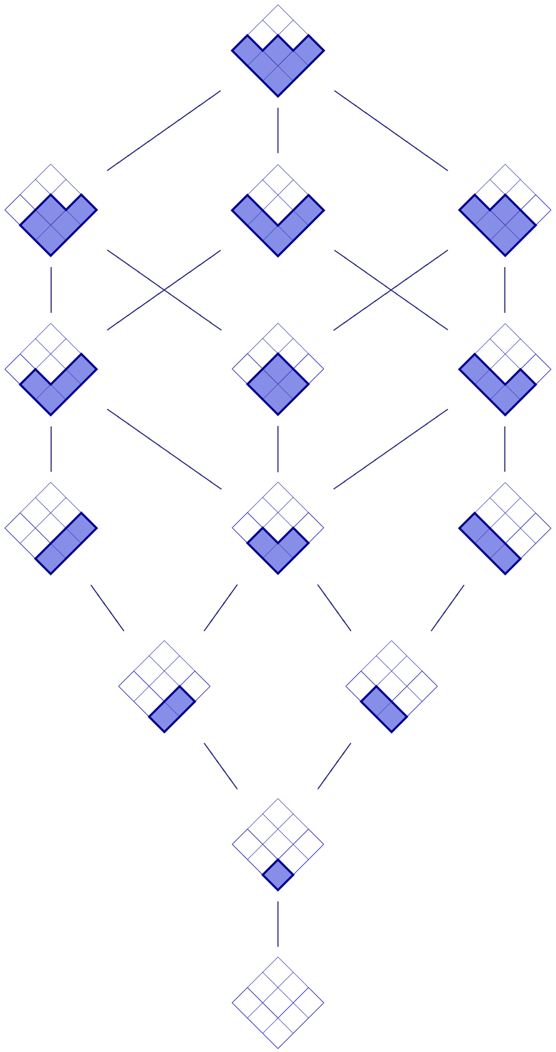
Example. $\lambda = (3, 2, 2, 1, 1) \subseteq (4^5)$.

$\lambda =$ 

path(λ) =  = UDDUDDUDU

We denote the set of all integer partitions by $\mathcal{P}$. It is well known that $\mathcal{P}$, partially ordered by set inclusion, is a lattice (the *Young lattice*).

Sublattice
of all partitions
$\lambda \subseteq (3, 2, 1)$:

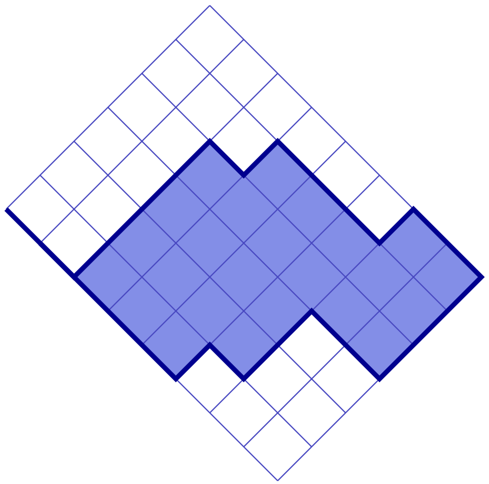


Let $\lambda, \mu \in \mathcal{P}$, with $\mu \subseteq \lambda$. Then we call $\lambda \setminus \mu$ a *skew partition*.

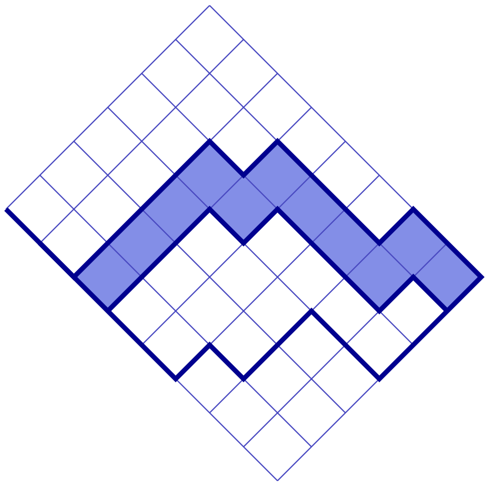
A skew partition is a *border strip* (also called a *ribbon*) if it contains no 2×2 square of cells. For brevity, we call a connected (by which we mean "rookwise connected") border strip a *cbs*.

The *outer border strip* θ of $\lambda \setminus \mu$ is the set of cells of $\lambda \setminus \mu$ such that the cell directly above it is not in $\lambda \setminus \mu$.

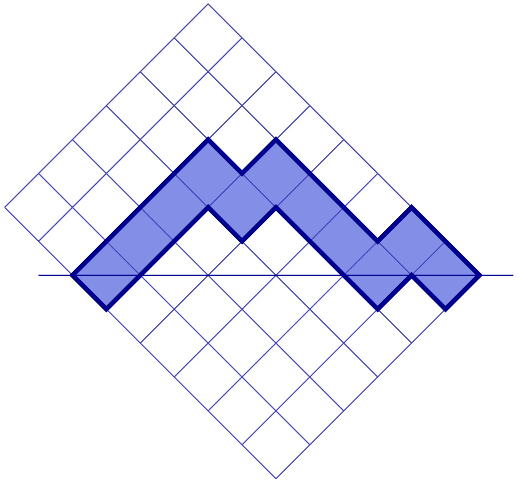
skew partition $\lambda \setminus \mu$



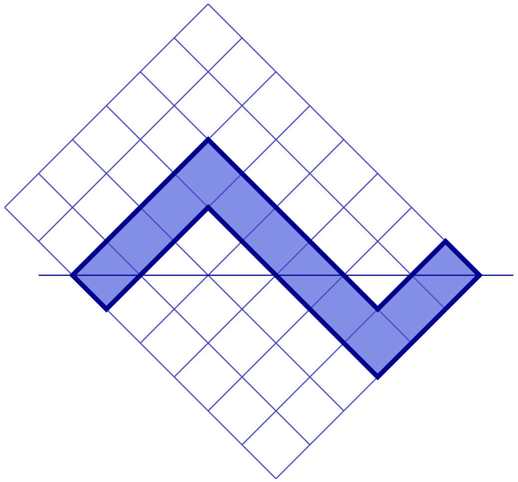
outer border strip of $\lambda \setminus \mu$



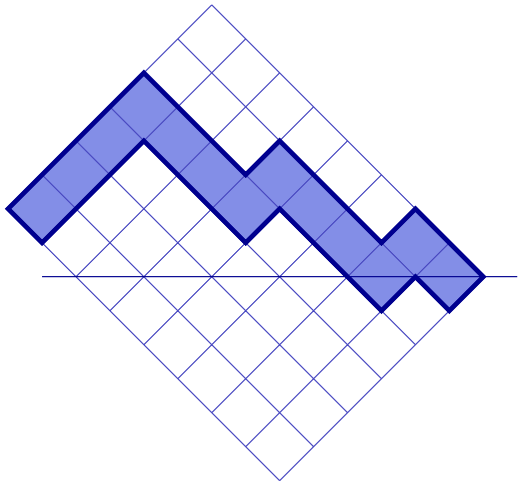
A cbs $\theta \subset \mathbf{P}^2$ is called a *Dyck cbs* if it is a "Dyck path", which means that no cell of θ has level strictly less than that of either the leftmost or the rightmost of its cells. (In particular, in a Dyck cbs the leftmost and rightmost cells have the same level.)



Dyck



non-Dyck



non-Dyck

Let $\lambda \setminus \mu \subset \mathbf{P}^2$ be a skew partition.

Recall that $\lambda \setminus \mu$ is defined to be *Dyck* in the following inductive way:

(1) the empty partition is *Dyck*,

(2) if $\lambda \setminus \mu$ is connected, then $\lambda \setminus \mu$ is *Dyck* if and only if
 (a) its outer border strip θ is a Dyck cbs,
 (b) $(\lambda \setminus \mu) \setminus \theta$ is Dyck,

(3) if $\lambda \setminus \mu$ is not connected, then $\lambda \setminus \mu$ is *Dyck* if and only if all of its connected components are Dyck.

Let $\lambda \setminus \mu \subset \mathbf{P}^2$ be a skew partition (not necessarily Dyck).

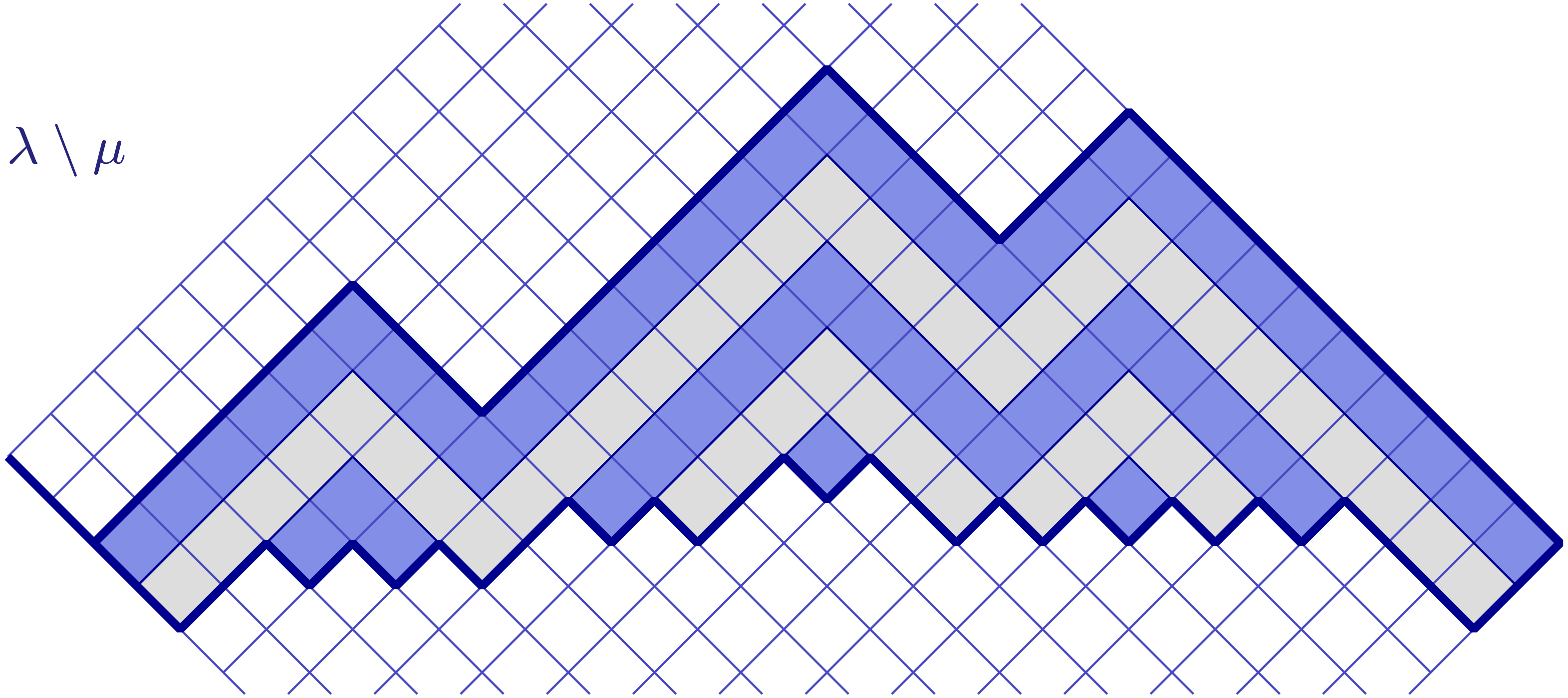
The *depth* of $\lambda \setminus \mu$ is defined inductively by

$$\mathrm{dp}(\lambda \setminus \mu) = \begin{cases} 0, & \text{if } \lambda = \mu, \\ c(\theta) + \mathrm{dp}((\lambda \setminus \mu) \setminus \theta), & \text{otherwise,} \end{cases}$$

where θ is the outer border strip of $\lambda \setminus \mu$ and

$$c(\theta) = \#\ \text{connected components of } \theta.$$

Example. Dyck skew partition:



$$\mathrm{dp}(\lambda \setminus \mu) = 8.$$

2. Parabolic Kazhdan-Lusztig polynomials

Theorem. (Deodhar, 1987) Let (W, S) be any Coxeter system and let $J \subseteq S$. Then, there is a unique family of polynomials

$$\{P^J_{u,v}(q)\}_{u,v \in W^J} \subseteq \mathbf{Z}[q]$$

such that, for all $u, v \in W^J$, with $u \leqslant v$, and fixed $s \in D(v)$, one has

$$P^J_{u,v}(q) = \widetilde{P}(q) - \sum_{\{u \leqslant w \leqslant vs : ws < w\}} \mu(w, vs) q^{\frac{\ell(w,v)}{2}} P^J_{u,w}(q),$$

where

$$\widetilde{P}(q) = \begin{cases} P^J_{us,vs}(q) + qP^J_{u,vs}(q), & \text{if } us < u, \\ qP^J_{us,vs}(q) + P^J_{u,vs}(q)\ , & \text{if } u < us \in W^J, \\ 0, & \text{if } u < us \notin W^J. \end{cases}$$

and

$$\mu(u, v) = \left[q^{\frac{\ell(u,v)-1}{2}} \right] (P^J_{u,v}).$$

The $P^J_{u,v}(q)$ are the *parabolic Kazhdan-Lusztig polynomials* of W^J.

For $J = \emptyset$, we get the (*ordinary*) *Kazhdan-Lusztig polynomials* of W:

$$P_{u,v}(q) = P^{\emptyset}_{u,v}(q).$$

Conversely, parabolic Kazhdan-Lusztig polynomials can be expressed in terms their ordinary counterparts.

Proposition. Let $J \subseteq S$, and $u, v \in W^J$. Then

$$P^J_{u,v}(q) = \sum_{w \in W_J} (-1)^{\ell(w)} P_{wu,v}(q).$$

The previous result has two interesting consequences.

Corollary. Let $I \subseteq J \subseteq S$, and $u, v \in W^J$. Then

$$P^J_{u,v}(q) = \sum_{w \in (W_J)^I} (-1)^{\ell(w)} P^I_{wu,v}(q).$$

Therefore, knowledge of the parabolic Kazhdan-Lusztig polynomials for a given $I \subseteq S$ implies knowledge of them for any J containing I.

Corollary. Let $J \subseteq S$, and $u, v \in W^J$. Then

$$\left[q^{\frac{\ell(u,v)-1}{2}} \right] (P_{u,v}(q)) = \left[q^{\frac{\ell(u,v)-1}{2}} \right] (P^J_{u,v}(q)).$$

Therefore knowledge of the parabolic Kazhdan-Lusztig polynomials for a given $J \subseteq S$ implies knowledge of the maximum-degree coefficient of the ordinary Kazhdan-Lusztig polynomials for all elements of W^J.

These are the coefficients that are of interest in the construction of the Kazhdan-Lusztig cells and representations.

Besides their connections with Kazhdan-Lusztig polynomials (which have applications in several areas of mathematics, including geometry of Schubert varieties and representation theory), the parabolic ones also play a direct role in the following areas:

- generalized Verma modules
- tilting modules
- quantized Schur algebras
- representation theory of the Lie algebra $\mathfrak{gl}_n$
- Macdonald polynomials
- partial flag varieties.

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- W. Soergel, *Kazhdan-Lusztig polynomials and a combinatoric for tilting modules*, Represent. Theory **1** (1997), 83-114.
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- M. Varagnolo, E. Vasserot, *On the decomposition matrices of the quantized Schur algebra*, Duke Math. J. **100** (1999), 267-297.
- B. Leclerc, J.-Y. Thibon, *Littlewood-Richard-son coefficients and Kazhdan-Lusztig polynomials*, Adv. Studies Pure Math. **28** (2000), 155-220.
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- M. Kashiwara, T. Tanisaki, *Parabolic Kazhdan-Lusztig polynomials and Schubert varieties*, J. Algebra **249** (2002), 306-325.

In [*Pacific Journal of Mathematics* **207** (2002), 257–286], Brenti found a closed formula for the parabolic Kazhdan-Lusztig polynomials for the *maximal quotients* of the symmetric group.

Theorem. (Brenti, 2002) Let $u, v \in S_n^{[n-1]\setminus\{i\}}$, with

$$\Lambda(v) = \lambda \quad \text{and} \quad \Lambda(u) = \mu.$$

Then

$$P^J_{u,v}(q) = \begin{cases} q^{\frac{|\lambda\setminus\mu| - \mathrm{dp}(\lambda\setminus\mu)}{2}}, & \text{if } \lambda \setminus \mu \text{ is Dyck,} \\ 0, & \text{otherwise.} \end{cases}$$

In this talk we generalize this result to the *quasi-minuscule* quotients.

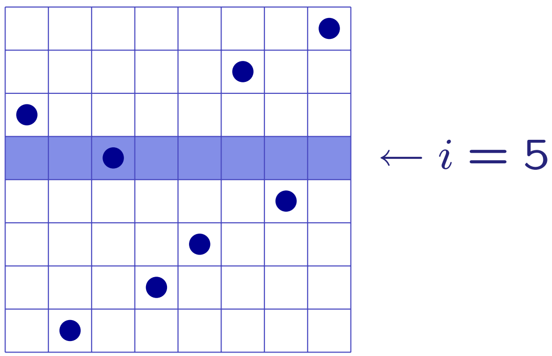
3. Quasi-minuscule quotiens

We will now give a combinatorial description of the quasi-minuscule quotients in S_n. We start with the following simple observation.

A permutation $v \in S_n$ belongs to $S_n^{[n-1]\setminus\{i-1,i\}}$ if and only if

$$v^{-1}(1) < \cdots < v^{-1}(i-1) \quad \text{and} \quad v^{-1}(i) < \cdots < v^{-1}(n).$$

Example. $v = 61523748 \in S_8^{[7]\setminus\{4,5\}}$.



Let $\lambda \subseteq (n^m)$ be a partition and let

$$\text{path}(\lambda) = x_1 \ldots x_{n+m}, \quad x_k \in \{\text{U}, \text{D}\}.$$

We say that an index $k \in [n+m-1]$ is a

$$\begin{cases} \textit{valley} \text{ of } \lambda, & \text{if } (x_k, x_{k+1}) = (\text{D}, \text{U}), \\ \textit{peak} \text{ of } \lambda, & \text{if } (x_k, x_{k+1}) = (\text{U}, \text{D}). \end{cases}$$

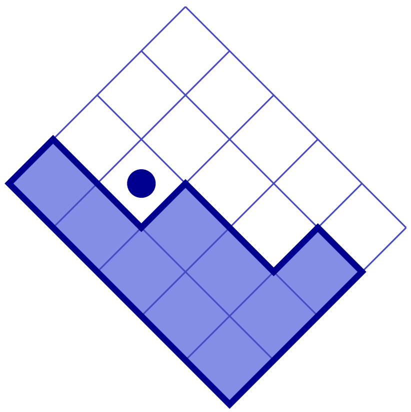
Definition. A *rooted partition* is a pair (λ, r), where λ is a partition with at least one valley and r is one of its valleys.

We think of a rooted partition as a lattice path with a ball in one of its valleys. If $\lambda \subseteq (n^m)$ and $\text{path}(\lambda) = x_1 \ldots, x_{n+m}$, then we set

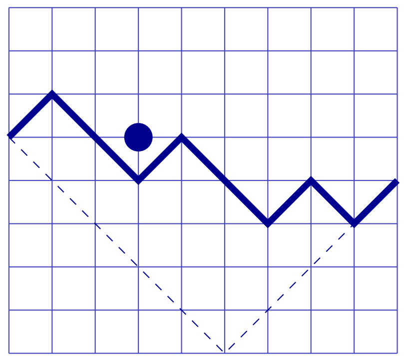
$$\text{path}(\lambda, r) = x_1 \ldots x_r \bullet x_{r+1} \ldots x_{n+m}$$

Example. $\lambda = (3,2,2,1,1) \subseteq (4^5)$ and $r = 3$.

$(\lambda, r) =$



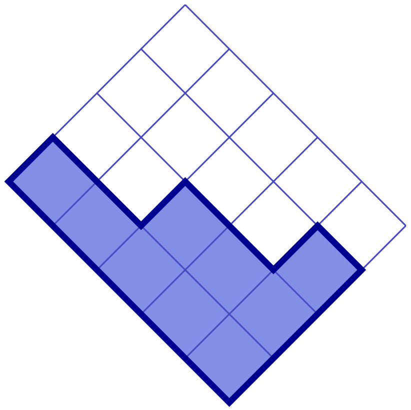
$\mathrm{path}(\lambda, r) =$



$=$ UDD $\bullet$ UDDUDU

Let $v \in S_n^{[n-1]\setminus\{i-1,i\}}$. The *partition* associated with v, denoted by $\Lambda(v)$, is the non-increasing rearrangement of the inversion table of v.

Example. $v = 61523748 \in S_8^{[7]\setminus\{4,5\}}$. Then

$$\Lambda(v) = (3,2,2,1,1) =$$ 

Remark. $\Lambda(v) \subseteq ((n-i+1)^i)$ and $v^{-1}(i)$ is a valley of $\Lambda(v)$.

Proposition. The map $v \mapsto (\Lambda(v), v^{-1}(i))$ is a bijection

$$S_n^{[n-1]\setminus\{i-1,i\}} \longleftrightarrow \{\text{rooted partitions } \subseteq ((n-i+1)^i)\}.$$

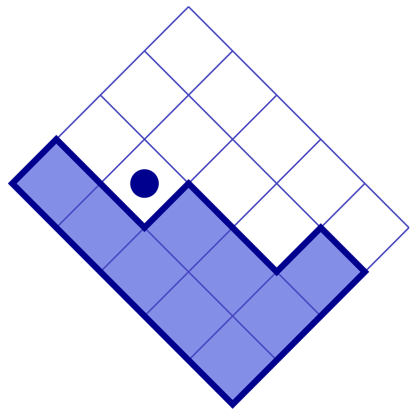
Furthermore, $\ell(v) = |\Lambda(v)|$.

The *rooted partition* associated with v is

$$\Lambda^\bullet(v) = (\Lambda(v), v^{-1}(i)).$$

Example. $v = 61523748 \in S_8^{[7]\setminus\{4,5\}}$. Then

$$\Lambda^\bullet(v) = ((3,2,2,1,1),3) =$$



The rooted partition $\Lambda^\bullet(v)$ can be constructed directly from v.

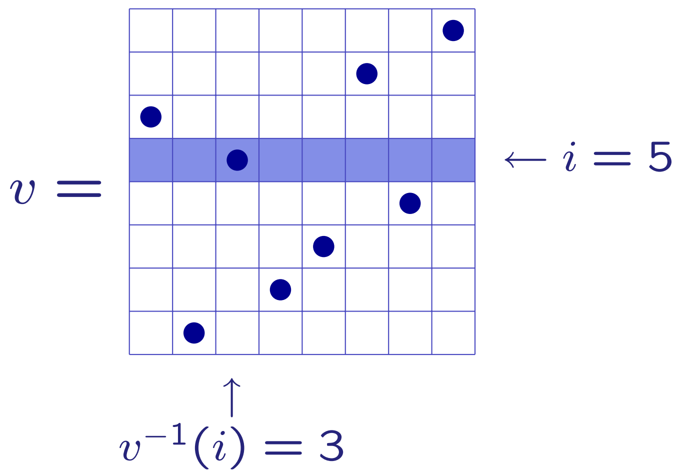
Proposition. Let $v \in S_n^{[n-1]\setminus\{i-1,i\}}$. Then

$$\mathsf{path}(\Lambda^\bullet(v)) = x_1 x_2 \ldots x_n,$$

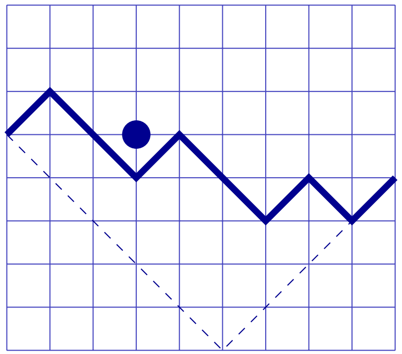
where

$$x_k = \begin{cases} \mathsf{D}, & \text{if } v(k) < i, \\ \mathsf{D} \bullet \mathsf{U}, & \text{if } v(k) = i, \\ \mathsf{U}, & \text{if } v(k) > i. \end{cases}$$

Example. $v = 61523748 \in S_8^{[7]\setminus\{4,5\}}$.



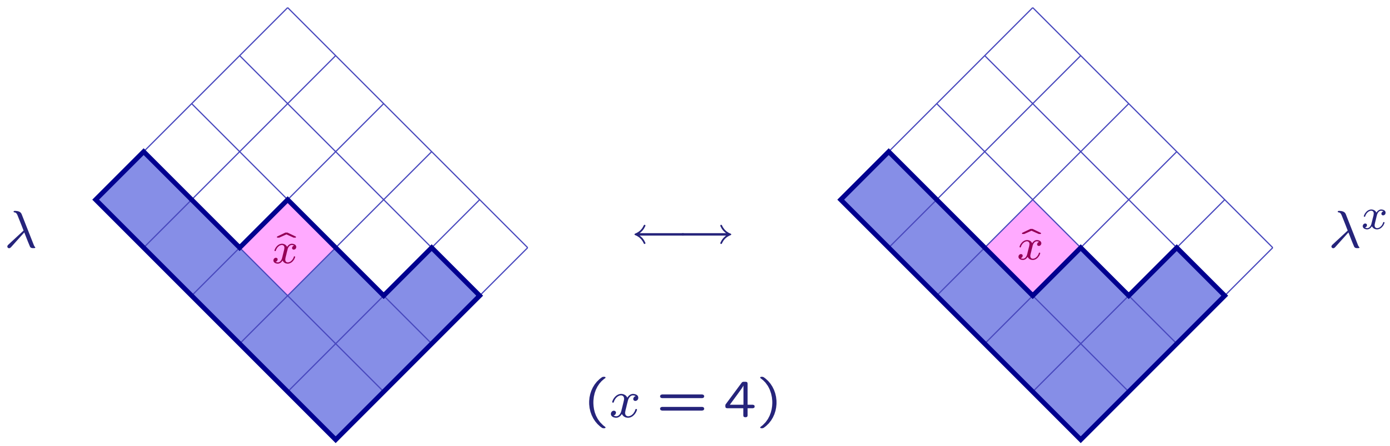
$\mathrm{path}(\lambda, r) =$



$= \underset{x_1}{\mathsf{U}} \ \underset{x_2}{\mathsf{D}} \ \underbrace{\mathsf{D} \bullet \mathsf{U}}_{x_3} \ \underset{x_4}{\mathsf{D}} \ \underset{x_5}{\mathsf{D}} \ \underset{x_6}{\mathsf{U}} \ \underset{x_7}{\mathsf{D}} \ \underset{x_8}{\mathsf{U}}$

Let λ be a partition. If x is a peak or a valley of λ, we denote by $\widehat{x}$ the cell immediately below x or above x, respectively. Then we set

$$\lambda^x = \begin{cases} \lambda \setminus \{\widehat{x}\}, & \text{if } x \text{ is a peak of } \lambda, \\ \lambda \cup \{\widehat{x}\}, & \text{if } x \text{ is a valley of } \lambda. \end{cases}$$



$(x = 4)$

The operator $(\,\cdot\,)^x$ is clearly an involution.

We now give a description of the Bruhat order on $S_n^{[n-1]\setminus\{i-1,i\}}$ in terms of rooted partitions, showing that, basically, the behaviour of the root is that of a ball subject to gravity.

Let (λ, r) be a rooted partition and let x be a valley of λ, such that λ^x has at least one valley. We say that (λ', r') is obtained from (λ, r) by an *elementary move* if $\lambda' = \lambda^x$ and

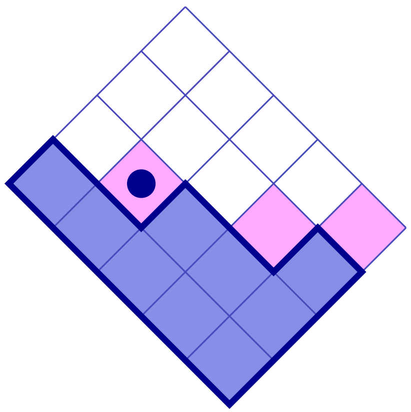
$$r' = \begin{cases} r, & \text{if } x \neq r, \\ \text{one of the valleys around its peak } x, & \text{if } x = r. \end{cases}$$

Proposition. Let $u, v \in S_n^{[n-1]\setminus\{i-1,i\}}$, with

$$\Lambda^\bullet(v) = (\lambda, r) \quad \text{and} \quad \Lambda^\bullet(u) = (\mu, t).$$

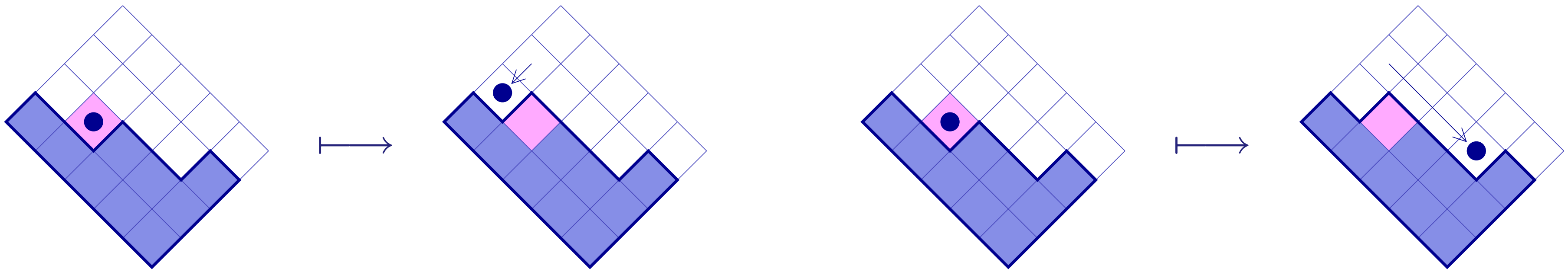
Then v covers u (in the Bruhat order) if and only if (λ, r) is obtained from (μ, t) by an elementary move.

Example. $v = 61523748 \in S_8^{[7]\setminus\{4,5\}}$.

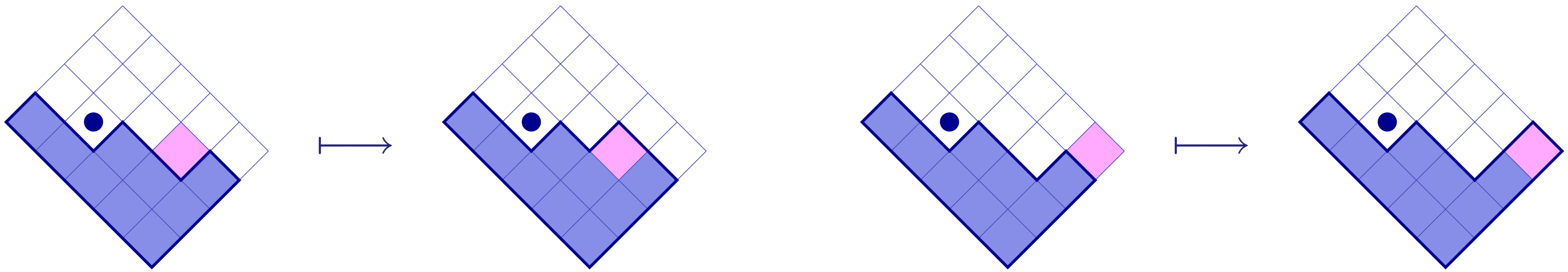
$\Lambda^\bullet(v) =$ 

$\text{valleys}(\Lambda^\bullet(v)) = \{3, 6, 8\}$.

Thus, there are four $w \in S_8^{[7]\backslash\{4,5\}}$ that cover v, obtained as follows:



$\Lambda^\bullet(v)$ $\Lambda^\bullet(w)$ $\Lambda^\bullet(v)$ $\Lambda^\bullet(w)$



$\Lambda^\bullet(v)$ $\Lambda^\bullet(w)$ $\Lambda^\bullet(v)$ $\Lambda^\bullet(w)$

The characterization of the covering relation implies the following.

Proposition. There is a bijection

$$\text{rooted partitions} \quad \longleftrightarrow \quad \text{covering relations in Young's lattice.}$$

Proposition. Let $u, v \in S_n^{[n-1]\setminus\{i-1,i\}}$. Then

$$u \leqslant v \quad \Longrightarrow \quad \Lambda(u) \subseteq \Lambda(v).$$

Note that the converse of the last assertion is not true in general.

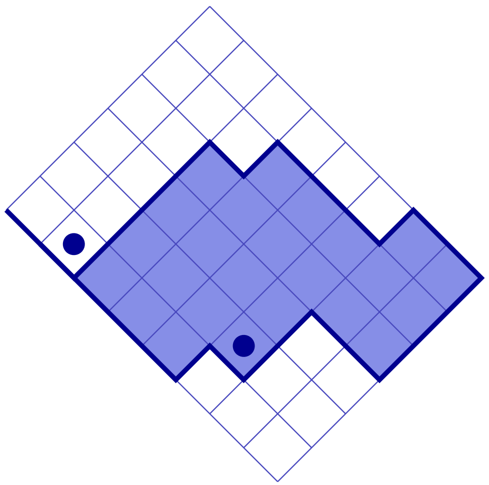
Example. $u = 16273548,\ v = 61523748 \in S_8^{[7]\setminus\{4,5\}}$.

$$\Lambda(u) = (3,2,2,1,0) \subseteq (3,2,2,1,1) = \Lambda(v), \quad \text{but } u \not\leqslant v.$$

4. $\bullet$-Dyck partitions

This is the main new combinatorial concept arising from this work.

If (λ, r) and (μ, t) are two rooted partitions such that $\mu \subseteq \lambda$, then we call $(\lambda, r) \setminus (\mu, t)$ a *skew rooted partition*.



Definition. A skew rooted partition $(\lambda, r) \setminus (\mu, t)$ is $\bullet$*-Dyck* if

(1) there are no peaks of λ strictly between the two roots,

(2) at least one of $\lambda \setminus \mu$ and $\lambda \setminus \mu^t$ is Dyck.

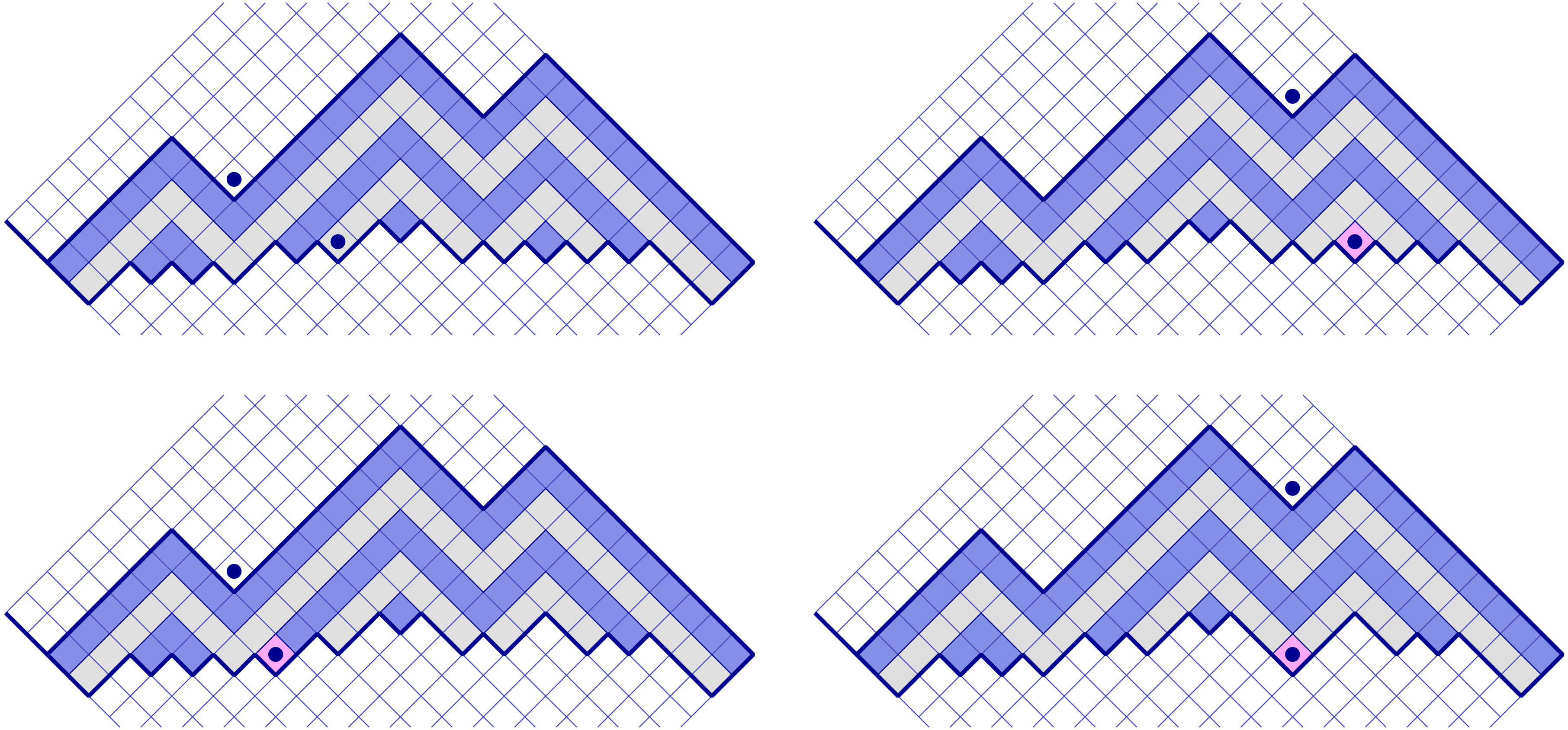
Let $(\lambda, r) \setminus (\mu, t)$ be $\bullet$-Dyck. The *depth* of $(\lambda, r) \setminus (\mu, t)$ is

$$\mathrm{dp}((\lambda, r) \setminus (\mu, t)) = \begin{cases} \mathrm{dp}(\lambda \setminus \mu), & \text{if } \lambda \setminus \mu \text{ is Dyck,} \\ \mathrm{dp}(\lambda \setminus \mu^t) + 1, & \text{if } \lambda \setminus \mu^t \text{ is Dyck,} \end{cases}$$

Proposition. Let $\lambda \setminus \mu$ be skew partition and let t be a valley of μ. Suppose that at least one of $\lambda \setminus \mu$ and $\lambda \setminus \mu^t$ is Dyck. Then $\lambda \setminus \mu$ and $\lambda \setminus \mu^t$ are both Dyck if and only if t is a peak of λ. In this case,

$$\mathrm{dp}(\lambda \setminus \mu) = \mathrm{dp}(\lambda \setminus \mu^t) + 1.$$

Four •-Dyck skew rooted partitions:



For all of them,

$$|\lambda \setminus \mu| = 98 \quad \text{and} \quad \mathrm{dp}((\lambda, r) \setminus (\mu, t)) = 8.$$

5. Main result

Theorem. (Brenti, I., Marietti, 2008) Let $u, v \in S_n^{[n-1]\setminus\{i-1,i\}}$, with

$$\Lambda^{\bullet}(v) = (\lambda, r) \quad \text{and} \quad \Lambda^{\bullet}(u) = (\mu, t).$$

Then

$$P^J_{u,v}(q) = \begin{cases} q^{\frac{|\lambda\setminus\mu| - \mathrm{dp}((\lambda,r)\setminus(\mu,t))}{2}}, & \text{if } (\lambda, r) \setminus (\mu, t) \text{ is } \bullet\text{-Dyck}, \\ 0, & \text{otherwise.} \end{cases}$$

Example. If $(\lambda, r) \setminus (\mu, t)$ is one of the previous four, then

$$P^J_{u,v}(q) = q^{\frac{98-8}{2}} = q^{45}.$$

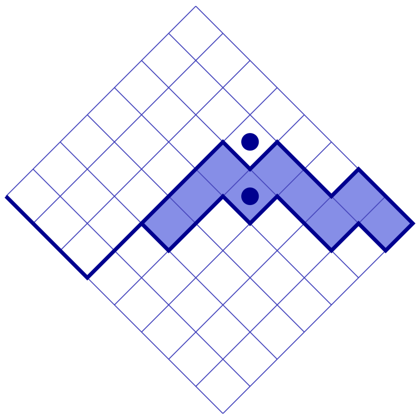
Corollary. Let $u, v \in S_n^{[n-1]\setminus\{i-1,i\}}$, with

$$\Lambda^\bullet(v) = (\lambda, r) \quad \text{and} \quad \Lambda^\bullet(u) = (\mu, t).$$

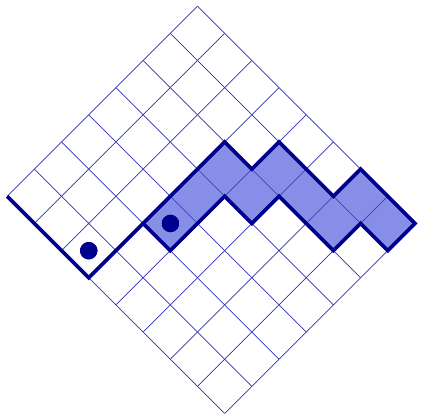
Then

$$\mu(u,v) = \begin{cases} 1, & \text{if } \lambda \setminus \mu \text{ is a Dyck cbs and there are} \\ & \quad \text{no peaks of } \lambda \text{ strictly between } r \text{ and } t, \\ 0, & \text{otherwise.} \end{cases}$$

Example. $\mu(u, v) = 1$ if $(\lambda, r) \setminus (\mu, t)$ is, for instance,



or



Our main result implies the analog result for *maximal quotients*.

Corollary. (Brenti, 2002) Let $u, v \in S_n^{[n-1]\setminus\{i\}}$, with

$$\Lambda(v) = \lambda \quad \text{and} \quad \Lambda(u) = \mu.$$

Then

$$P_{u,v}^J(q) = \begin{cases} q^{\frac{|\lambda\setminus\mu| - \mathrm{dp}(\lambda\setminus\mu)}{2}}, & \text{if } \lambda \setminus \mu \text{ is Dyck,} \\ 0, & \text{otherwise.} \end{cases}$$

We now consider the quasi-minuscule quotient $S_n^{[n-1]\setminus\{1,n-1\}}$.

A permutation $v \in S_n$ belongs to $S_n^{[n-1]\setminus\{1,n-1\}}$ if and only if

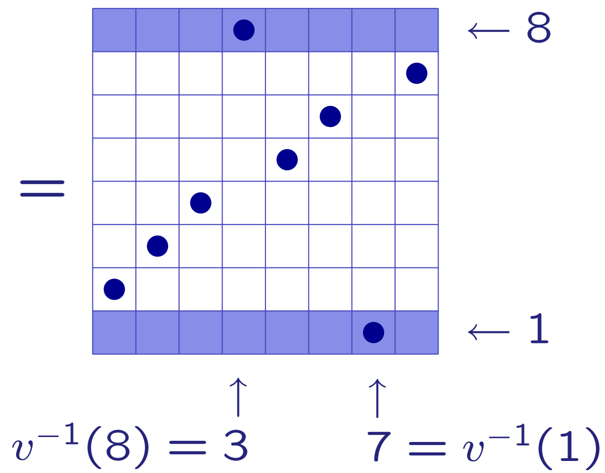
$$v^{-1}(2) < v^{-1}(3) < \cdots < v^{-1}(n-1).$$

Given $v \in S_n^{[n-1]\setminus\{i\}}$, we let

$$\Lambda_0(v) = (v^{-1}(1), v^{-1}(n)).$$

Example. $v = 23485617 \in S_8^{[7]\setminus\{1,7\}}$.

$v =$



$\Lambda_0(v) = (7, 3).$

Proposition. The map $v \mapsto \Lambda_0(v)$ is a bijection

$$S_n^{[n-1]\setminus\{i\}} \quad \longleftrightarrow \quad \{(a,b) \in [n]^2 : a \neq b\}.$$

Furthermore, if $\Lambda_0(u) = (a,b)$ and $\Lambda_0(v) = (c,d)$, then

$$u \leqslant v \quad \Longleftrightarrow \quad a \leqslant c \text{ and } b \geqslant d.$$

Theorem. (Brenti, I., Marietti, 2008) Let $u, v \in S_n^{[n-1]\setminus\{i\}}$, with

$$\Lambda_0(v) = (a,b) \quad \text{and} \quad \Lambda_0(u) = (c,d).$$

Then

$$P_{u,v}^J(q) = \begin{cases} q^{c-d-2}, & \text{if } a-1 \leqslant d \leqslant a \leqslant b \leqslant c \leqslant b+1, \\ 0, & \text{otherwise.} \end{cases}$$

6. Open problems

In [M. Kashiwara, T. Tanisaki, *J. Algebra*, **249** (2002), 306–325] a geometric interpretation of the parabolic Kazhdan-Lusztig polynomials for Weyl groups was given in terms of intersection homology.

In view of this, the following problem is natural.

Open problem. Find a geometric proof of our main theorem.

A geometric proof for the case of maximal quotients has been recently found in [N. Perrin, *Compositio Math.*, **143** (2007), 1255–1312].

The following *non-negativity conjecture* is well known.

Conjecture. (Kazhdan-Lusztig, 1979) Let W be any Coxeter group and $u, v \in W$. Then $P_{u,v}(q)$ has non-negative coefficients.

It is widely believed (although not stated anywhere in the literature) that the same non-negativity property holds for the *parabolic* Kazhdan-Lusztig polynomials.

Conjecture. Let (W, S) be any Coxeter system, $J \subseteq S$ and $u, v \in W^J$. Then $P^J_{u,v}(q)$ has non-negative coefficients.

It is true for Weyl groups by the above geometric interpretation.

The following is a recent conjecture by Brenti.

Conjecture. (Brenti, 2008) Let (W, S) be any Coxeter system and

$$I \subseteq J \subseteq S.$$

Then, for all $u, v \in W^J$,

$$P^I_{u,v}(q) \geqslant P^J_{u,v}(q)$$

(coefficientwise).

7. Enumerative results

7.1 Enumeration of Dyck partitions

Let $\lambda \subseteq (n^m)$ be a partition and consider the associated path

$$\text{path}(\lambda) = x_1 \ldots x_{n+m}, \quad x_k \in \{\mathsf{U}, \mathsf{D}\}.$$

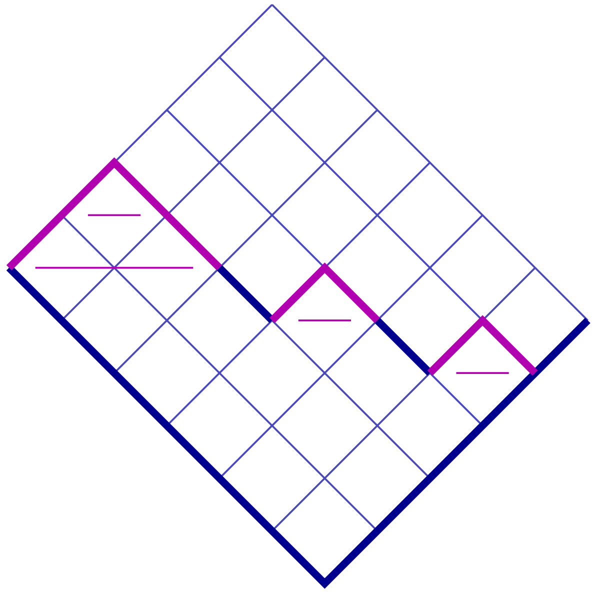
We make the substitution $\quad \mathsf{U} \longleftrightarrow (\qquad \mathsf{D} \longleftrightarrow).$

We define the *matching set* and the *matching number* of λ by

$$M(\lambda) = \{k \in [n+m] : \text{parenthesis } x_k \text{ is matched}\},$$

$$\text{mtc}(\lambda) = \frac{|M(\lambda)|}{2} = \# \text{ pairs of matched parentheses in path}(\lambda).$$

Example. $\lambda = (4,3,3,2,2,2) \subseteq (5^6)$.

$\lambda =$ 

$$\text{path}(\lambda) = \underline{(}\ \underline{(}\ \underline{)}\ \underline{)}\ )\ \underline{(}\ \underline{)}\ )\ \underline{(}\ \underline{)}\ ($$

$$M(\lambda) = \{1,2,3,4,6,7,10,11\}$$

$$\text{mtc}(\lambda) = 4$$

In 2002, Brenti enumerated the partitions μ contained in a given partition λ such that $\lambda \setminus \mu$ is Dyck and found a q-analog formula.

This is a reformulation of his result.

Theorem. (Brenti, 2002) Let $\lambda \subseteq (n^m)$. Then

$$|\{\mu \subseteq \lambda : \lambda \setminus \mu \text{ is Dyck}\} = 2^{\mathsf{mtc}(\lambda)}.$$

More generally, the following q-analog holds:

$$\sum_{\substack{\mu \subseteq \lambda \\ \lambda \setminus \mu \text{ is Dyck}}} q^{\mathsf{dp}(\lambda \setminus \mu)} = (q+1)^{\mathsf{mtc}(\lambda)}.$$

Recently, *all* the Dyck skew partition contained in a given rectangle have been enumerated and a q-analog has been found.

Theorem. (I., August 2008)

$$|\{\lambda\setminus\mu\subseteq(n^m)\text{ Dyck}\}| = \sum_{k=0}^{\min\{n,m\}} \frac{n+m-2k+1}{n+m-k+1}\binom{n+m}{k}2^k.$$

More generally, the following q-analog holds:

$$\sum_{\substack{\lambda\setminus\mu\subseteq(n^m)\\ \lambda\setminus\mu\text{ is Dyck}}} q^{\mathrm{dp}(\lambda\setminus\mu)} = \sum_{k=0}^{\min\{n,m\}} \frac{n+m-2k+1}{n+m-k+1}\binom{n+m}{k}(q+1)^k.$$

We have the following equivalent formulas.

Theorem. (I., August 2008)

$$|\{\lambda \setminus \mu \subseteq (n^m) \text{ Dyck}\}| = \binom{n+m}{n} 2^{\min\{n,m\}+1} - \sum_{k=0}^{\min\{n,m\}} \binom{n+m}{k} 2^k.$$

$$\sum_{\substack{\lambda\setminus\mu\subseteq(n^m)\\ \lambda\setminus\mu \text{ is Dyck}}} q^{\mathrm{dp}(\lambda\setminus\mu)} = \binom{n+m}{n}(q+1)^{\min\{n,m\}+1} - \sum_{k=0}^{\min\{n,m\}} \binom{n+m}{k}(q+1)^k$$

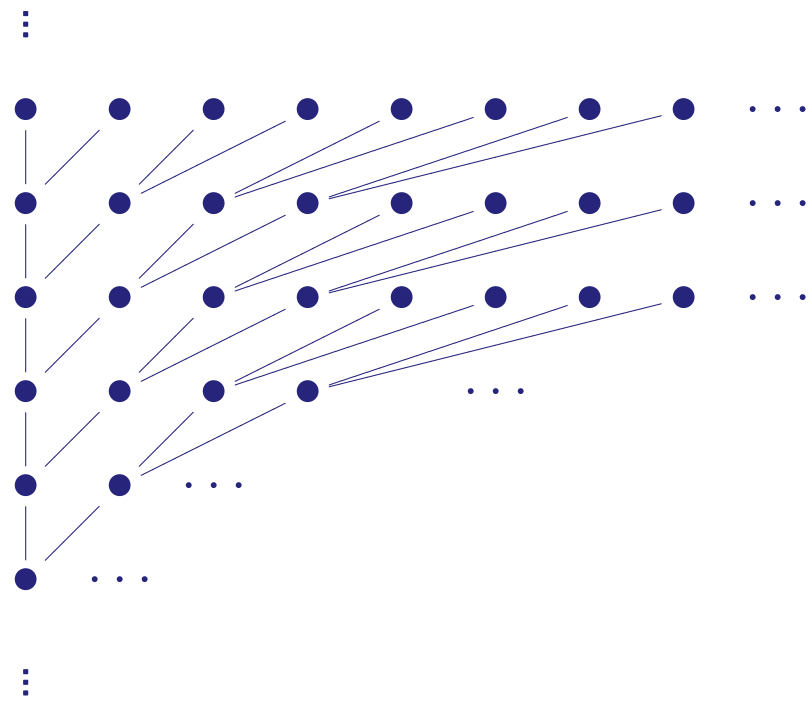
$$= \binom{n+m}{n}(q+1)^{\min\{n,m\}+1} - L_{\min\{n,m\}}((q+2)^{n+m}).$$

Where L_h is the *truncating operator*: $L_h\left(\sum_{k=0}^{n} a_k q^k\right) = \sum_{k=0}^{h} a_k q^k.$

7.2 Connection with paths on regular trees

For any integer $d \geqslant 2$, we denote by T_d the *d-regular tree*, that is the (infinite) tree where all the vertices have degree d.

T_3



Given two vertices x and y in a graph G, we denote by $\mathsf{Paths}_{G,\ell}(x,y)$ the set of all paths in G of length ℓ from x to y.

Theorem. (I., August 2008) Let $n, m \in \mathbf{P}$.

Let x, y be two vertices of T_3 at distance $|n-m|$. Then

$$|\{\lambda \setminus \mu \subseteq (n^m) : \lambda \setminus \mu \text{ is Dyck}\}| = |\mathsf{Paths}_{T_3,n+m}(x,y)|.$$

More generally, we have the following q-analog.

Let $q \in \mathbf{Z}_{\geqslant 0}$ and x, y be two vertices of T_{q+2} at distance $|n-m|$. Then

$$\sum_{\substack{\lambda\setminus\mu\subseteq(n^m)\\ \lambda\setminus\mu \text{ is Dyck}}} q^{\mathsf{dp}(\lambda\setminus\mu)} = |\mathsf{Paths}_{T_{q+2},n+m}(x,y)|.$$

For both results we gave combinatorial bijective proofs.

7.3 Enumeration of $\bullet$-Dyck partitions

Let (λ, r) be a rooted partition contained in (n^m), with

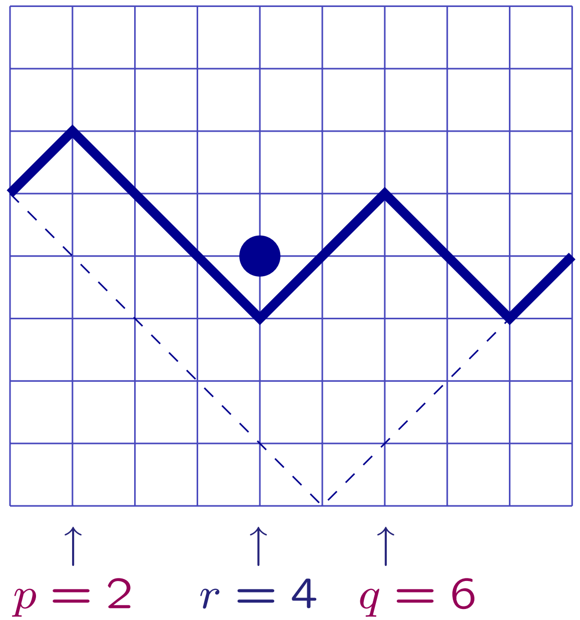
$$\text{path}(\lambda, r) = x_1 \dots x_r \bullet x_{r+1} \dots x_{n+m}, \quad x_k \in \{\text{D}, \text{U}\}.$$

Let p and q, with p minimal and q maximal, be such that

$$x_p \dots x_r \bullet x_{r+1} \dots x_q = \text{DD} \dots \text{D} \bullet \text{UU} \dots \text{U}.$$

In other words, $p-1$ is the first peak to the left of r (unless $p=1$) and q is the first peak to the right of r (unless $q = n+m$).

Example. $\lambda = (3, 3, 1, 1, 1) \subseteq (4^5)$ and $r = 4$.



Theorem. (I., August 2008) Let (λ, r) be a rooted partition and let p and q be as above. Then

$$|\{(\mu, t) : (\lambda, r) \setminus (\mu, t) \text{ is } \bullet\text{-Dyck}\}| = 2^{a-1}(b + 2^c - d),$$

where a, b, c, d only depend on λ, namely

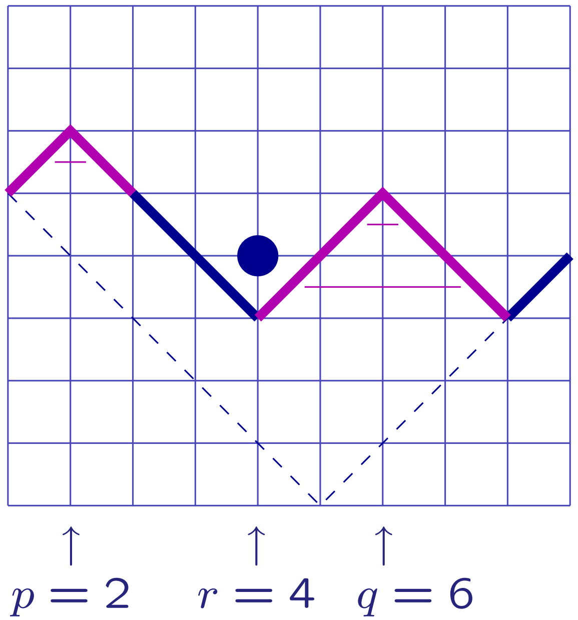
$$a = \mathrm{mtc}(\lambda),$$

$$b = |M(\lambda) \cap [p, q]|,$$

$$c = |M(\lambda) \cap \{r, r+1\}|,$$

$$d = |M(\lambda) \cap \{p, q\}|.$$

Example. $\lambda = (3,3,1,1,1) \subseteq (4^5)$ and $r = 4$.



$$a = \mathrm{mtc}(\lambda) = 3$$

$$b = |M(\lambda) \cap [p,q]| = 3$$

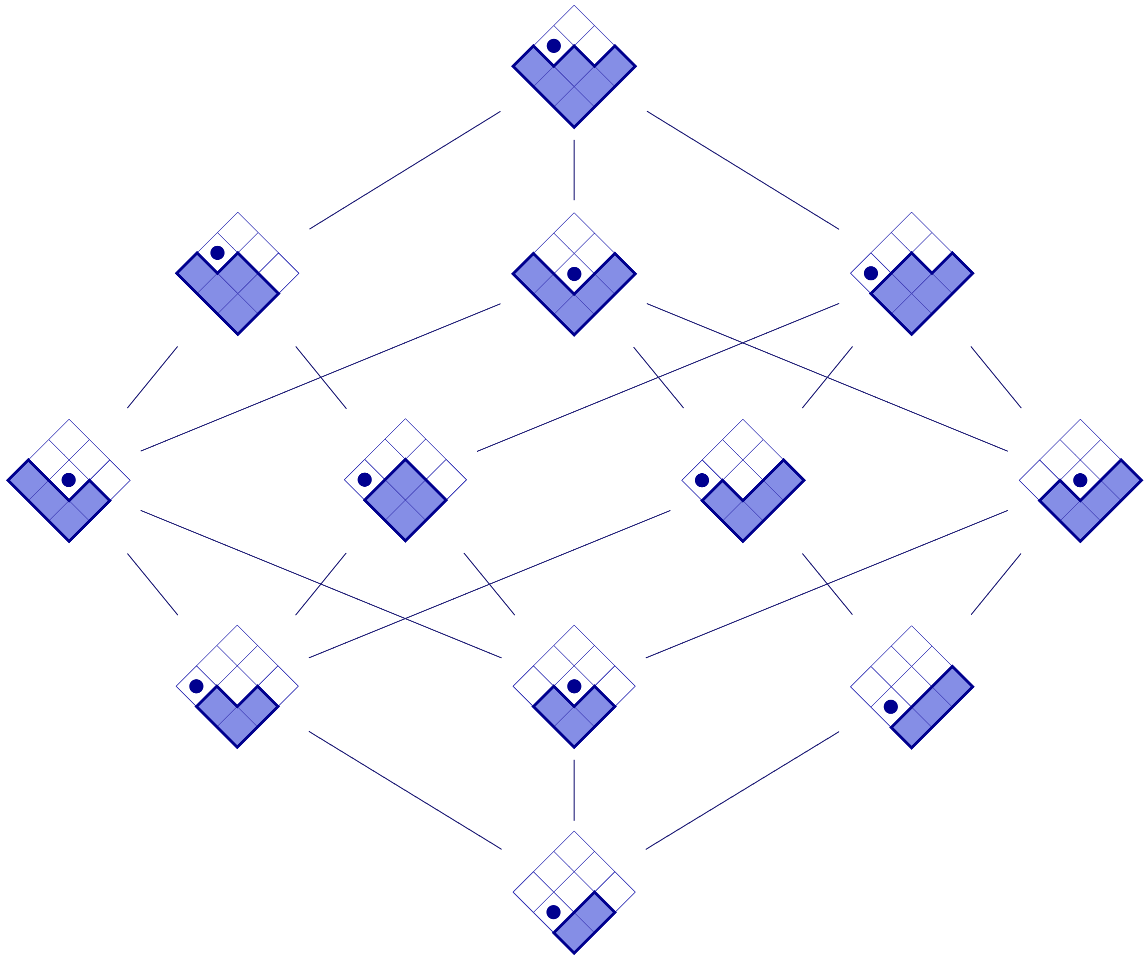
$$c = |M(\lambda) \cap \{r, r+1\}| = 1$$

$$d = |M(\lambda) \cap \{p,q\}| = 2$$

$$|\{(\mu,t) : (\lambda,r) \setminus (\mu,t) \text{ is } \bullet\text{-Dyck}\}| = 2^{3-1}(3 + 2^1 - 2) = 12.$$

Example. $\lambda = (3, 2, 1) \subseteq (3^3)$ and $r = 2$. Similarly,

$$|\{(\mu, t) : (\lambda, r) \setminus (\mu, t) \text{ is } \bullet\text{-Dyck}\}| = 12.$$



ありがとうございました

Thank you very much!