Dyck partitions, quasi-minuscule quotiens and Kazhdan-Lusztig polynomials

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partly based on a joint work with Francesco Brenti and Mario Marietti
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1. Background

1.1 Coxeter groups

$W$: Coxeter group  \quad $S$: set of generators

Set of \textit{reflections} of $W$:  \quad $T = \{vsv^{-1} : v \in W, \ s \in S\}$.

Let $v \in W$. The \textit{length} of $v$ is

$$\ell(v) = \min\{k : v \text{ is a product of } k \text{ generators}\}.$$  

The (right) \textit{descent set} of $v$ is

$$D(v) = \{s \in S : \ell(vs) < \ell(v)\}.$$
**Bruhat graph** of $W$: directed graph with $W$ as vertex set and

$$u \rightarrow v \iff u^{-1}v \in T \text{ and } \ell(u) < \ell(v).$$

**Bruhat order** of $W$: partial order on $W$ defined by

$$u \leq v \iff u = u_0 \rightarrow u_1 \rightarrow \cdots \rightarrow u_k = v.$$  

$W$, with the Bruhat order, is a graded poset with rank function $\ell$.

For $u, v \in W$, with $u < v$, we set

$$\ell(u, v) = \ell(v) - \ell(u) \text{ (distance in the Bruhat order).}$$
Let $J \subseteq S$ be a fixed subset of generators.

The *parabolic subgroup* of $W$ generated by $J$ is

$$W_J = \langle J \rangle.$$  

The *quotient* of $W$ by $J$ is

$$W^J = \{v \in W : \ell(sv) > \ell(v) \text{ for all } s \in J\}.$$  

We will consider particular quotients of the symmetric group.
1.2 The symmetric group

\( P = \{1, 2, 3, \ldots \}, \quad [n] = \{1, 2, \ldots, n\} \quad (n \in P), \)

**Symmetric group**: \( S_n = \{v : [n] \to [n] \text{ bijection}\}. \)

We denote \( v \in S_n \) by the word \( v(1) v(2) \ldots v(n) \) and by its *diagram*.

**Example.** \( v = 61523748 \in S_8 \) has diagram
$S_n$ is a Coxeter group, with generators the simple transpositions:

$$S = \{(1, 2), (2, 3), \ldots, (n - 1, n)\}.$$  

When we refer to these generators, the transposition $(i, i+1)$ is simply denoted by $i$. With this convention, the set of generators of $S_n$ is

$$S = [n - 1].$$

The reflections are all the transpositions:

$$T = \{(i, j) \in [n]^2 : i < j\}.$$
Let \( v \in S_n \). The length of \( v \) is the number of its inversions:

\[
\ell(v) = |\{(i, j) \in [n]^2 : i < j \text{ and } v(i) > v(j)\}|.
\]

The descent set of \( v \) is

\[
D(v) = |\{i \in [n-1] : v(i) > v(i + 1)\}|.
\]

Let \( J \subseteq [n-1] \). The quotient of \( S_n \) by \( J \) is

\[
(S_n)^J = \{v \in S_n : v^{-1}(r) < v^{-1}(r + 1) \text{ for all } r \in J\}.
\]
The **maximal quotients** of $S_n$ are obtained by taking

$$J = [n - 1] \setminus \{i\} \quad (i \in [n - 1]).$$

The **quasi-minuscule quotients** of $S_n$ are obtained by taking

$$J = [n - 1] \setminus \{i - 1, i\} \quad (2 \leq i \leq n - 1)$$

or

$$J = [n - 1] \setminus \{1, n - 1\}.$$

In this talk we study the quasi-minuscule quotients of $S_n$. 
1.3 Partitions and lattice paths

We identify a partition \( \lambda = (\lambda_1, \ldots, \lambda_k) \subseteq (n^m) \) with its diagram:

\[
\{(i, j) \in \mathbb{P}^2 : 1 \leq i \leq k \text{ and } 1 \leq j \leq \lambda_i\}.
\]

**Example.** \( \lambda = (3, 2, 2, 1, 1) \subseteq (4^5) \).
Given a partition $\lambda \subseteq (n^m)$, the path associated with $\lambda$ is the lattice path from $(0, m)$ to $(n + m, n)$, with steps $(1, 1)$ (up steps) and $(1, -1)$ (down steps) which is the upper border of the diagram of $\lambda$:

$$\text{path}(\lambda) = x_1 x_2 \ldots x_{n+m}, \quad \text{with } x_k \in \{U, D\},$$

Note that $\text{path}(\lambda)$ has exactly $n$ U’s and $m$ D’s.

**Example.** $\lambda = (3, 2, 2, 1, 1) \subseteq (4^5)$.
We denote the set of all integer partitions by $\mathcal{P}$. It is well known that $\mathcal{P}$, partially ordered by set inclusion, is a lattice (the *Young lattice*).

Sublattice of all partitions $\lambda \subseteq (3, 2, 1)$:
Let $\lambda, \mu \in \mathcal{P}$, with $\mu \subseteq \lambda$. Then we call $\lambda \setminus \mu$ a **skew partition**.

A skew partition is a **border strip** (also called a **ribbon**) if it contains no $2 \times 2$ square of cells. For brevity, we call a connected (by which we mean “rookwise connected”) border strip a **cbs**.

The **outer border strip** $\theta$ of $\lambda \setminus \mu$ is the set of cells of $\lambda \setminus \mu$ such that the cell directly above it is not in $\lambda \setminus \mu$. 
A cbs $\theta \subset \mathbb{P}^2$ is called a *Dyck cbs* if it is a “Dyck path”, which means that no cell of $\theta$ has level strictly less than that of either the leftmost or the rightmost of its cells. (In particular, in a Dyck cbs the leftmost and rightmost cells have the same level.)
Let $\lambda \setminus \mu \subset \text{P}^2$ be a skew partition.

Recall that $\lambda \setminus \mu$ is defined to be *Dyck* in the following inductive way:

1. The empty partition is *Dyck*.

2. If $\lambda \setminus \mu$ is connected, then $\lambda \setminus \mu$ is *Dyck* if and only if
   a. its outer border strip $\theta$ is a Dyck cbs,
   b. $(\lambda \setminus \mu) \setminus \theta$ is Dyck,

3. If $\lambda \setminus \mu$ is not connected, then $\lambda \setminus \mu$ is *Dyck* if and only if all of its connected components are Dyck.
Let $\lambda \setminus \mu \subset \mathbb{P}^2$ be a skew partition (not necessarily Dyck).

The depth of $\lambda \setminus \mu$ is defined inductively by

$$dp(\lambda \setminus \mu) = \begin{cases} 0, & \text{if } \lambda = \mu, \\ c(\theta) + dp((\lambda \setminus \mu) \setminus \theta), & \text{otherwise,} \end{cases}$$

where $\theta$ is the outer border strip of $\lambda \setminus \mu$ and

$$c(\theta) = \# \text{ connected components of } \theta.$$
Example. Dyck skew partition:

\[ \lambda \setminus \mu \]

\[ dp(\lambda \setminus \mu) = 8. \]
2. Parabolic Kazhdan-Lusztig polynomials

**Theorem.** (Deodhar, 1987) Let \((W, S)\) be any Coxeter system and let \(J \subseteq S\). Then, there is a unique family of polynomials
\[
\{P_{u,v}^J(q)\}_{u,v \in W_J} \subseteq \mathbb{Z}[q]
\]
such that, for all \(u, v \in W_J\), with \(u \leq v\), and fixed \(s \in D(v)\), one has
\[
P_{u,v}^J(q) = \tilde{P}(q) - \sum_{\{u \leq w \leq vs : ws < w\}} \mu(w, vs) q^{\frac{\ell(w,v)}{2}} P_{u,w}^J(q),
\]
where
\[
\tilde{P}(q) = \begin{cases} 
P_{us,vs}^J(q) + qP_{u,vs}^J(q), & \text{if } us < u, \\
pP_{us,vs}^J(q) + P_{u,vs}^J(q), & \text{if } u < us \in W^J, \\
0, & \text{if } u < us \notin W^J.
\end{cases}
\]
and
\[
\mu(u, v) = \left[ q^{\frac{\ell(u,v)-1}{2}} \right] (P_{u,v}^J).
\]
The $P_{u,v}^J(q)$ are the \textit{parabolic Kazhdan-Lusztig polynomials} of $W^J$.

For $J = \emptyset$, we get the \textit{(ordinary) Kazhdan-Lusztig polynomials} of $W$:

$$P_{u,v}(q) = P_{u,v}^\emptyset(q).$$

Conversely, parabolic Kazhdan-Lusztig polynomials can be expressed in terms their ordinary counterparts.

**Proposition.** Let $J \subseteq S$, and $u, v \in W^J$. Then

$$P_{u,v}^J(q) = \sum_{w \in W_J} (-1)^{\ell(w)} P_{wu,v}(q).$$

The previous result has two interesting consequences.
Corollary. Let $I \subseteq J \subseteq S$, and $u, v \in W^J$. Then

$$P_{u,v}^J(q) = \sum_{w \in (W_J)^I} (-1)^{\ell(w)} P_{wu,v}^I(q).$$

Therefore, knowledge of the parabolic Kazhdan-Lusztig polynomials for a given $I \subseteq S$ implies knowledge of them for any $J$ containing $I$.

Corollary. Let $J \subseteq S$, and $u, v \in W^J$. Then

$$\left[ q^{\frac{\ell(u,v)-1}{2}} \right] (P_{u,v}(q)) = \left[ q^{\frac{\ell(u,v)-1}{2}} \right] (P_{u,v}^J(q)).$$

Therefore knowledge of the parabolic Kazhdan-Lusztig polynomials for a given $J \subseteq S$ implies knowledge of the maximum-degree coefficient of the ordinary Kazhdan-Lusztig polynomials for all elements of $W^J$.

These are the coefficients that are of interest in the construction of the Kazhdan-Lusztig cells and representations.
Besides their connections with Kazhdan-Lusztig polynomials (which have applications in several areas of mathematics, including geometry of Schubert varieties and representation theory), the parabolic ones also play a direct role in the following areas:

- generalized Verma modules
- tilting modules
- quantized Schur algebras
- representation theory of the Lie algebra $\mathfrak{gl}_n$
- Macdonald polynomials
- partial flag varieties.


In [Pacific Journal of Mathematics 207 (2002), 257–286], Brenti found a closed formula for the parabolic Kazhdan-Lusztig polynomials for the maximal quotients of the symmetric group.

**Theorem.** (Brenti, 2002) Let \( u, v \in S_{n-1} \setminus \{i\} \), with

\[
\Lambda(v) = \lambda \quad \text{and} \quad \Lambda(u) = \mu.
\]

Then

\[
P_{u,v}^J(q) = \begin{cases} 
q^{\frac{|\lambda \setminus \mu| - dp(\lambda \setminus \mu)}{2}}, & \text{if } \lambda \setminus \mu \text{ is Dyck,} \\
0, & \text{otherwise.}
\end{cases}
\]

In this talk we generalize this result to the quasi-minuscule quotients.
3. Quasi-minuscule quotients

We will now give a combinatorial description of the quasi-minuscule quotients in $S_n$. We start with the following simple observation.

A permutation $v \in S_n$ belongs to $S_n^{[n-1]\{i-1,i\}}$ if and only if

$$v^{-1}(1) < \cdots < v^{-1}(i-1) \quad \text{and} \quad v^{-1}(i) < \cdots < v^{-1}(n).$$

Example. $v = 61523748 \in S_8^{[7]\{4,5\}}$.
Let $\lambda \subseteq (n^m)$ be a partition and let

$$\text{path}(\lambda) = x_1 \ldots x_{n+m}, \quad x_k \in \{U, D\}.$$ 

We say that an index $k \in [n + m - 1]$ is a

$$\begin{align*}
\text{valley of } \lambda, & \quad \text{if } (x_k, x_{k+1}) = (D, U), \\
\text{peak of } \lambda, & \quad \text{if } (x_k, x_{k+1}) = (U, D).
\end{align*}$$

**Definition.** A **rooted partition** is a pair $(\lambda, r)$, where $\lambda$ is a partition with at least one valley and $r$ is one of its valleys.

We think of a rooted partition as a lattice path with a ball in one of its valleys. If $\lambda \subseteq (n^m)$ and $\text{path}(\lambda) = x_1 \ldots, x_{n+m}$, then we set

$$\text{path}(\lambda, r) = x_1 \ldots x_r \bullet x_{r+1} \ldots x_{n+m}$$
Example. \( \lambda = (3, 2, 2, 1, 1) \subseteq (4^5) \) and \( r = 3 \).

\[
(\lambda, r) = \begin{array}{c}
\end{array}
\]

\[
\text{path}(\lambda, r) = UDD \bullet UDDUDU
\]
Let $v \in S_{n}^{[n-1]\{i-1,i\}}$. The partition associated with $v$, denoted by $\Lambda(v)$, is the non-increasing rearrangement of the inversion table of $v$.

**Example.** $v = 61523748 \in S_{8}^{[7]\{4,5\}}$. Then

\[
\Lambda(v) = (3, 2, 2, 1, 1) = \begin{array}{c}
\end{array}
\]

**Remark.** $\Lambda(v) \subseteq ((n - i + 1)^{i})$ and $v^{-1}(i)$ is a valley of $\Lambda(v)$. 
Proposition. The map \( v \mapsto (\Lambda(v), v^{-1}(i)) \) is a bijection

\[
S_{n-1}^{\{i-1,i\}} \leftrightarrow \{\text{rooted partitions} \subseteq ((n - i + 1)^i)\}.
\]

Furthermore, \( \ell(v) = |\Lambda(v)|. \)

The *rooted partition* associated with \( v \) is

\[
\Lambda^\bullet(v) = (\Lambda(v), v^{-1}(i)).
\]

Example. \( v = 61523748 \in S_8^{[7]\{4,5\}}. \) Then

\[
\Lambda^\bullet(v) = ((3, 2, 2, 1, 1), 3) = \begin{array}{c}
\end{array}
\]
The rooted partition $\Lambda^\bullet(v)$ can be constructed directly from $v$.

**Proposition.** Let $v \in S_{n-1}\backslash\{i-1,i\}$. Then

$$\text{path}(\Lambda^\bullet(v)) = x_1x_2\ldots x_n,$$

where

$$x_k = \begin{cases} D, & \text{if } v(k) < i, \\ D \cdot U, & \text{if } v(k) = i, \\ U, & \text{if } v(k) > i. \end{cases}$$
Example. \( v = 61523748 \in S_8^{[7]\setminus\{4,5\}} \).
Let $\lambda$ be a partition. If $x$ is a peak or a valley of $\lambda$, we denote by $\hat{x}$ the cell immediately below $x$ or above $x$, respectively. Then we set

$$
\lambda^x = \begin{cases} 
\lambda \setminus \{\hat{x}\}, & \text{if } x \text{ is a peak of } \lambda, \\
\lambda \cup \{\hat{x}\}, & \text{if } x \text{ is a valley of } \lambda.
\end{cases}
$$

The operator $(\cdot)^x$ is clearly an involution.
We now give a description of the Bruhat order on $S_{n-1}\setminus\{i-1,i\}$ in terms of rooted partitions, showing that, basically, the behaviour of the root is that of a ball subject to gravity.

Let $(\lambda, r)$ be a rooted partition and let $x$ be a valley of $\lambda$, such that $\lambda^x$ has at least one valley. We say that $(\lambda', r')$ is obtained from $(\lambda, r)$ by an *elementary move* if $\lambda' = \lambda^x$ and

$$r' = \begin{cases} r, & \text{if } x \neq r, \\ \text{one of the valleys around its peak } x, & \text{if } x = r. \end{cases}$$
Proposition. Let \( u, v \in S_n^{\{n-1\}\{i-1,i\}} \), with
\[
\Lambda^\bullet(v) = (\lambda, r) \quad \text{and} \quad \Lambda^\bullet(u) = (\mu, t).
\]
Then \( v \) covers \( u \) (in the Bruhat order) if and only if \( (\lambda, r) \) is obtained from \( (\mu, t) \) by an elementary move.

Example. \( v = 61523748 \in S_8^{\{7\}\{4,5\}} \).

\[
\Lambda^\bullet(v) = \text{valleys}(\Lambda^\bullet(v)) = \{3, 6, 8\}.
\]
Thus, there are four $w \in S_8^{[7]\{4,5\}}$ that cover $v$, obtained as follows:
The characterization of the covering relation implies the following.

**Proposition.** There is a bijection

\[
\text{rooted partitions} \leftrightarrow \text{covering relations in Young's lattice.}
\]

**Proposition.** Let \( u, v \in S_{n-1}^{\{i-1,i\}} \). Then

\[
u \leq v \implies \Lambda(u) \subseteq \Lambda(v).
\]

Note that the converse of the last assertion is not true in general.

**Example.** \( u = 16273548, \ v = 61523748 \in S_8^{\{4,5\}} \).

\[
\Lambda(u) = (3, 2, 2, 1, 0) \subseteq (3, 2, 2, 1, 1) = \Lambda(v), \quad \text{but } u \not\leq v.
\]
4. \(-\text{Dyck partitions}\)

This is the main new combinatorial concept arising from this work. If \((\lambda, r)\) and \((\mu, t)\) are two rooted partitions such that \(\mu \subseteq \lambda\), then we call \((\lambda, r) \setminus (\mu, t)\) a \textit{skew rooted partition}.
Definition. A skew rooted partition \((\lambda, r) \backslash (\mu, t)\) is \(-Dyck\) if

1. there are no peaks of \(\lambda\) strictly between the two roots,
2. at least one of \(\lambda \backslash \mu\) and \(\lambda \backslash \mu^t\) is Dyck.

Let \((\lambda, r) \backslash (\mu, t)\) be \(-Dyck\). The depth of \((\lambda, r) \backslash (\mu, t)\) is

\[
dp((\lambda, r) \backslash (\mu, t)) = \begin{cases} 
  dp(\lambda \backslash \mu), & \text{if } \lambda \backslash \mu \text{ is Dyck}, \\
  dp(\lambda \backslash \mu^t) + 1, & \text{if } \lambda \backslash \mu^t \text{ is Dyck}, 
\end{cases}
\]

Proposition. Let \(\lambda \backslash \mu\) be skew partition and let \(t\) be a valley of \(\mu\). Suppose that at least one of \(\lambda \backslash \mu\) and \(\lambda \backslash \mu^t\) is Dyck. Then \(\lambda \backslash \mu\) and \(\lambda \backslash \mu^t\) are both Dyck if and only if \(t\) is a peak of \(\lambda\). In this case,

\[
dp(\lambda \backslash \mu) = dp(\lambda \backslash \mu^t) + 1.
\]
Four \bullet-Dyck skew rooted partitions:

For all of them,

$$|\lambda \setminus \mu| = 98 \quad \text{and} \quad \text{dp}((\lambda, r) \setminus (\mu, t)) = 8.$$
5. Main result

**Theorem.** (Brenti, I., Marietti, 2008) Let \( u, v \in S_n^{[n-1]\{i-1,i\}} \), with
\[
\Lambda^\bullet(v) = (\lambda, r) \quad \text{and} \quad \Lambda^\bullet(u) = (\mu, t).
\]
Then
\[
P_{u,v}^J(q) = \begin{cases} 
  q^{\frac{|\lambda \setminus \mu| - dp((\lambda, r) \setminus (\mu, t))}{2}}, & \text{if} \ (\lambda, r) \setminus (\mu, t) \text{ is } \bullet\text{-Dyck,} \\
  0, & \text{otherwise.}
\end{cases}
\]

**Example.** If \((\lambda, r) \setminus (\mu, t)\) is one of the previous four, then
\[
P_{u,v}^J(q) = q^{\frac{98-8}{2}} = q^{45}.
\]
Corollary. Let \( u, v \in S^{[n-1]\setminus\{i-1,i\}} \), with
\[
\Lambda \circ (v) = (\lambda, r) \quad \text{and} \quad \Lambda \circ (u) = (\mu, t).
\]
Then
\[
\mu(u, v) = \begin{cases} 1, & \text{if } \lambda \setminus \mu \text{ is a Dyck cbs and there are no peaks of } \lambda \text{ strictly between } r \text{ and } t, \\ 0, & \text{otherwise.} \end{cases}
\]

Example. \( \mu(u, v) = 1 \) if \( (\lambda, r) \setminus (\mu, t) \) is, for instance,
Our main result implies the analog result for maximal quotients.

**Corollary.** (Brenti, 2002) Let $u, v \in S_{n-1}\{i\}$, with
\[\Lambda(v) = \lambda \quad \text{and} \quad \Lambda(u) = \mu.\]
Then
\[P_{u,v}(q) = \begin{cases} 
q^{\frac{\left|\lambda \setminus \mu\right| - dp(\lambda \setminus \mu)}{2}}, & \text{if } \lambda \setminus \mu \text{ is Dyck}, \\
0, & \text{otherwise}.
\end{cases}\]
We now consider the quasi-minusculc quotient $S_{n-1}\{1,n-1\}$.

A permutation $v \in S_n$ belongs to $S_{n-1}\{1,n-1\}$ if and only if

$$v^{-1}(2) < v^{-1}(3) < \cdots < v^{-1}(n-1).$$

Given $v \in S_{n-1}\{i\}$, we let

$$\Lambda_0(v) = (v^{-1}(1), v^{-1}(n)).$$

Example. $v = 23485617 \in S_8^{[7]\{1,7\}}$.

$$v = \begin{array}{cccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\end{array}
$$

$v^{-1}(8) = 3$  \quad  7 = v^{-1}(1)$

$\Lambda_0(v) = (7, 3)$. 

Proposition. The map $v \mapsto \Lambda_0(v)$ is a bijection

$$S_n^{[n-1]\{i\}} \longleftrightarrow \{(a, b) \in [n]^2 : a \neq b\}.$$ 

Furthermore, if $\Lambda_0(u) = (a, b)$ and $\Lambda_0(v) = (c, d)$, then

$$u \leq v \iff a \leq c \text{ and } b \geq d.$$ 

Theorem. (Brenti, I., Marietti, 2008) Let $u, v \in S_n^{[n-1]\{i\}}$, with

$$\Lambda_0(v) = (a, b) \text{ and } \Lambda_0(u) = (c, d).$$

Then

$$P_{u,v}^J(q) = \begin{cases} 
q^{c-d-2}, & \text{if } a - 1 \leq d \leq a \leq b \leq c \leq b + 1, \\
0, & \text{otherwise}.
\end{cases}$$
6. Open problems


In view of this, the following problem is natural.

**Open problem.** Find a geometric proof of our main theorem.

A geometric proof for the case of maximal quotients has been recently found in [N. Perrin, *Compositio Math.*, 143 (2007), 1255–1312].
The following *non-negativity conjecture* is well known.

**Conjecture.** (Kazhdan-Lusztig, 1979) Let $W$ be any Coxeter group and $u, v \in W$. Then $P_{u,v}(q)$ has non-negative coefficients.

It is widely believed (although not stated anywhere in the literature) that the same non-negativity property holds for the *parabolic* Kazhdan-Lusztig polynomials.

**Conjecture.** Let $(W, S)$ be any Coxeter system, $J \subseteq S$ and $u, v \in W^J$. Then $P^J_{u,v}(q)$ has non-negative coefficients.

It is true for Weyl groups by the above geometric interpretation.
The following is a recent conjecture by Brenti.

**Conjecture.** (Brenti, 2008) Let \((W, S)\) be any Coxeter system and 
\[ I \subseteq J \subseteq S. \]

Then, for all \(u, v \in W^J\),
\[ P^I_{u,v}(q) \geq P^J_{u,v}(q) \]
(coefficientwise).
7. Enumerative results

7.1 Enumeration of Dyck partitions

Let \( \lambda \subseteq (n^m) \) be a partition and consider the associated path

\[
\text{path}(\lambda) = x_1 \ldots x_{n+m}, \quad x_k \in \{U, D\}.
\]

We make the substitution \( U \leftrightarrow ( \quad D \leftrightarrow ) \).

We define the \textit{matching set} and the \textit{matching number} of \( \lambda \) by

\[
M(\lambda) = \{ k \in [n + m] : \text{parenthesis } x_k \text{ is matched} \},
\]

\[
\text{mtc}(\lambda) = \frac{|M(\lambda)|}{2} = \# \text{ pairs of matched parentheses in } \text{path}(\lambda).
\]
Example. $\lambda = (4, 3, 3, 2, 2, 2) \subseteq (5^6)$.

\[ \lambda = \begin{array}{c}
\text{path}(\lambda) = ( ( ) \ ) ( ) ( ) ( ) ( ) ( ) \\
M(\lambda) = \{1, 2, 3, 4, 6, 7, 10, 11\} \\
\text{mtc}(\lambda) = 4
\end{array} \]
In 2002, Brenti enumerated the partitions $\mu$ contained in a given partition $\lambda$ such that $\lambda \setminus \mu$ is Dyck and found a $q$-analog formula. This is a reformulation of his result.

**Theorem.** (Brenti, 2002) Let $\lambda \subseteq (n^m)$. Then

$$|\{\mu \subseteq \lambda : \lambda \setminus \mu \text{ is Dyck}\}| = 2^{mtc(\lambda)}.$$

More generally, the following $q$-analog holds:

$$\sum_{\mu \subseteq \lambda} q^{dp(\lambda \setminus \mu)} = (q + 1)^{mtc(\lambda)}.$$
Recently, \textit{all} the Dyck skew partition contained in a given rectangle have been enumerated and a $q$-analog has been found.

**Theorem.** (I., August 2008)

$$|\{\lambda \setminus \mu \subseteq (n^m) \text{ Dyck}\}| = \min\{n, m\} \sum_{k=0}^{\min\{n, m\}} \frac{n + m - 2k + 1}{n + m - k + 1} \binom{n + m}{k} 2^k.$$

More generally, the following $q$-analog holds:

$$\sum_{\substack{\lambda \setminus \mu \subseteq (n^m) \\ \lambda \setminus \mu \text{ is Dyck}}} q^{\text{dp}(\lambda \setminus \mu)} = \min\{n, m\} \sum_{k=0}^{\min\{n, m\}} \frac{n + m - 2k + 1}{n + m - k + 1} \binom{n + m}{k} (q + 1)^k.$$
We have the following equivalent formulas.

**Theorem.** (I., August 2008)

\[
|\{\lambda \setminus \mu \subseteq (n^m) \text{ Dyck}\}| = \binom{n + m}{n} 2^\min\{n,m\} + 1 - \sum_{k=0}^{\min\{n,m\}} \binom{n + m}{k} 2^k.
\]

\[
\sum_{\lambda \setminus \mu \subseteq (n^m) \atop \lambda \setminus \mu \text{ is Dyck}} q^{\mathrm{dp}(\lambda \setminus \mu)} = \binom{n + m}{n}(q + 1)^\min\{n,m\} + 1 - \sum_{k=0}^{\min\{n,m\}} \binom{n + m}{k}(q + 1)^k
\]

\[
= \binom{n + m}{n}(q + 1)^\min\{n,m\} + 1 - L_{\min\{n,m\}}((q + 2)^{n+m}).
\]

Where \(L_h\) is the **truncating operator**: \(L_h \left( \sum_{k=0}^{n} a_kq^k \right) = \sum_{k=0}^{h} a_kq^k\).
7.2 Connection with paths on regular trees

For any integer $d \geq 2$, we denote by $T_d$ the $d$-regular tree, that is the (infinite) tree where all the vertices have degree $d$. 

![Diagram of $T_3$]
Given two vertices \( x \) and \( y \) in a graph \( G \), we denote by \( \text{Paths}_{G,\ell}(x,y) \) the set of all paths in \( G \) of length \( \ell \) from \( x \) to \( y \).

**Theorem.** (I., August 2008) Let \( n, m \in \mathbb{P} \).

Let \( x, y \) be two vertices of \( T_3 \) at distance \(|n - m|\). Then

\[
|\{\lambda \setminus \mu \subseteq (nm) : \lambda \setminus \mu \text{ is Dyck}\}| = |\text{Paths}_{T_3,n+m}(x,y)|.
\]

More generally, we have the following \( q \)-analog.

Let \( q \in \mathbb{Z}_{\geq 0} \) and \( x, y \) be two vertices of \( T_{q+2} \) at distance \(|n - m|\). Then

\[
\sum_{\substack{\lambda \setminus \mu \subseteq (nm) \\ \lambda \setminus \mu \text{ is Dyck}}} q^{\text{dp}(\lambda \setminus \mu)} = |\text{Paths}_{T_{q+2},n+m}(x,y)|.
\]

For both results we gave combinatorial bijective proofs.
7.3 Enumeration of $\bullet$-Dyck partitions

Let $(\lambda, r)$ be a rooted partition contained in $(n^m)$, with

$$\text{path}(\lambda, r) = x_1 \cdots x_r \bullet x_{r+1} \cdots x_{n+m}, \quad x_k \in \{D, U\}.$$ 

Let $p$ and $q$, with $p$ minimal and $q$ maximal, be such that

$$x_p \cdots x_r \bullet x_{r+1} \cdots x_q = DD \cdots D \bullet UU \cdots U.$$ 

In other words, $p - 1$ is the first peak to the left of $r$ (unless $p = 1$) and $q$ is the first peak to the right of $r$ (unless $q = n + m$).
Example. \( \lambda = (3, 3, 1, 1, 1) \subseteq (4^5) \) and \( r = 4 \).

\[ p = 2 \quad r = 4 \quad q = 6 \]
**Theorem.** (I., August 2008) Let $(\lambda, r)$ be a rooted partition and let $p$ and $q$ be as above. Then

\[ |\{(\mu, t) : (\lambda, r) \setminus (\mu, t) \text{ is } \bullet \text{-Dyck}\}| = 2^{a-1}(b + 2^c - d), \]

where $a, b, c, d$ only depend on $\lambda$, namely

- $a = \text{mtc}(\lambda)$,
- $b = |M(\lambda) \cap [p, q]|$,
- $c = |M(\lambda) \cap \{r, r + 1\}|$,
- $d = |M(\lambda) \cap \{p, q\}|$. 
Example. \( \lambda = (3, 3, 1, 1, 1) \subseteq (4^5) \) and \( r = 4 \).

\[
\begin{align*}
a &= \text{mtc}(\lambda) = 3 \\
b &= |M(\lambda) \cap [p, q]| = 3 \\
c &= |M(\lambda) \cap \{r, r + 1\}| = 1 \\
d &= |M(\lambda) \cap \{p, q\}| = 2
\end{align*}
\]

\[
|\{(\mu, t) : (\lambda, r) \setminus (\mu, t) \text{ is } \bullet\text{-Dyck}\}| = 2^{3-1}(3 + 2^1 - 2) = 12.
\]
Example. \( \lambda = (3, 2, 1) \subseteq (3^3) \) and \( r = 2 \). Similarly,
\[
|\left\{ (\mu, t) : (\lambda, r) \setminus (\mu, t) \text{ is } \bullet\text{-Dyck} \right\}| = 12.
\]
ありがとうございます

Thank you very much!