GEOMETRIC DERIVED HALL ALGEBRA

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Abstract. We give a geometric formulation of Toën's derived Hall algebra by constructing Grothendieck's six operations for the derived category of lisse-étale constructible sheaves on the derived stacks of complexes. Our formulation is based on an variant of Laszlo and Olsson's theory of derived categories and six operations for algebraic stacks. We also give an ∞-theoretic explanation of the theory of derived stacks, which was originally constructed by Toën and Vezzosi in terms of model theoretical language.

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0. Introduction

0.1. The *derived Hall algebra* introduced by Toën [T1] is a version of Ringel-Hall algebra, and roughly speaking it is a "Hall algebra for complexes". In the case of the ordinary Ringel-Hall algebra for the abelian category of representations of a quiver, Lusztig [Lus] established a geometric formulation using the theory of perverse sheaves in the equivariant derived category of ℓ-adic constructible sheaves on the moduli spaces of the representations, and his formulation gives rise to the theory of canonical bases for quantum groups. As mentioned in [T1, §1, Related and future works], it is natural to expect a similar geometric formulation for derived Hall algebras using the moduli space of *complexes* of representations.

At the moment when [T1] appears, the theory of *derived stacks* was just being under construction, which would realize the moduli space of complexes. Soon after, Toën and Vezzosi [TVe1, TVe2] completed the works on derived stacks, and based on them Toën and Vaquié [TVa] constructed the moduli space of dg-modules. In [TVa, §0.6], it was announced among several stuffs that a geometric formulation of derived Hall algebra was being studied. As far as we understand, it has not appeared yet.

The purpose of this article is to give such a geometric formulation of derived Hall algebras. More precisely speaking, we want to give the following materials.

- The theory of derived categories of *lisse-étale constructible sheaves* on derived stacks and the construction of *Grothendieck's six operations* on them.
- The theory of *perverse ℓ-adic constructible sheaves* on derived stacks.
- The geometric formulation of derived Hall algebra.

Date: December 10, 2019.

Any such attempt will require *derived algebraic geometry*. Although the works [TVe1, TVe2, TVa] are based on the language of *model categories* so that it might be natural to work over such a language, we decided to work using the language of *∞-categories* following Lurie's derived algebraic geometry [Lur5] – [Lur14]. The reason is that today most papers on derived algebraic geometry uses the language of ∞-categories rather than that of model categories. A "byproduct" of our project is

- A "translation" of model-categorical [TVe1, TVe2, TVa] into the language of ∞-categories.

Such a translation is in fact a trivial one, and it seems that experts on derived algebraic geometry use both languages freely. We decided to give somewhat a textbook-like explanation on this point, and we will start the main text with some recollection on the theory of ∞-categories and ∞-topoi (§1). The theory of derived stacks in the sense of Toën and Vezzosi will be explained in the language of ∞-categories (§2).

We will construct the derived category of sheaves on derived stacks using the theory of *stable ∞-category* established by Lurie [Lur2]. For this purpose, we will give in §3 an exposition of the general theory of the ∞-category of sheaves on ∞-topoi and the associated stable ∞-category. There are many overlaps between our exposition and a (very small) part of Lurie's derived algebraic geometry [Lur5]–[Lur14]. We decided to give a rather self-contained presentation for the completeness of this article.

The main object of this article is the *lisse-étale sheaves* on derived stacks, which will be introduced in §5. Our definition is a simple analogue of the lisse-étale sheaves on the algebraic stacks established by Laumon and Moret-Bailly [LM]. In the case of algebraic stacks, the lisse-étale topos is constructed using the étale topos on algebraic spaces. Thus we need a derived analogue of algebraic spaces, which we call *derived algebraic spaces* and introduce in §4.

In [LO1, LO2], Laszlo and Orson constructed the Grothendieck six operations on the derived category of constructible sheaves on algebraic stacks, based on the correction [O1] of a technical error on the lisse-étale topos in [LM]. In the sequel [LO3] they also give the theory of perverse ℓ-adic sheaves and weights on them over algebraic stacks. Our construction of six operations (§6, §7) and definition of perverse ℓ-adic sheaves (§8) are just a copy of their argument.

Fundamental materials will be established up to §8, and we will then turn to the derived Hall algebra. In §9 we review Toën and Vaquié's construction [TVa] of the moduli space of dg-modules over a dg-category via derived stack. In §10 we explain the definition of derived Hall albebras, and give its geometric formulation, the main purpose of this article.

Let us sketch an outline of the construction here. Let D be a locally finite dg-category over $\mathbb{F}_q$ (see §9 for an account on dg-categories).

Fact ([TVa])**.** We have the *moduli stack* $\mathcal{P}\mathrm{f}(\mathrm{D})$ *of perfect dg-modules* over D^{op}. It is a derived stack, locally geometric and locally of finite presentation.

We can also construct the moduli stack of cofibrations $X \to Y$ of perfect-modules over D^{op}, denoted by $\mathcal{E}(\mathrm{D})$. Then there exist morphisms

$$s, c, t : \mathcal{E}(\mathrm{D}) \longrightarrow \mathcal{P}\mathrm{f}(\mathrm{D})$$

of derived stacks which send $u : X \to Y$ to

$$s(u) = X, \quad c(u) = Y, \quad t(u) = Y \coprod^{X} 0.$$

where s, t are smooth and c is proper. Thus we have a square

$$\begin{array}{ccc} \mathcal{E}(\mathrm{D}) & \xrightarrow{\ c\ } & \mathcal{P}\mathrm{f}(\mathrm{D}) \\ {\scriptstyle p:=s\times t}\downarrow & & \\ \mathcal{P}\mathrm{f}(\mathrm{D}) \times \mathcal{P}\mathrm{f}(\mathrm{D}) & & \end{array}$$

of derived stacks with smooth p and proper c.

Next let $\Lambda := \overline{\mathbb{Q}}_\ell$ be the field of ℓ-adic numbers where ℓ and q are assumed to be coprime. Then we have the *derived category* $\mathrm{D}^b_c(\mathcal{X}, \Lambda)$ *of constructible lisse-étale Λ-sheaves* over a locally geometric derived stack $\mathcal{X}$. We also have derived functors (§4).

Applying the general theory to the present situation, we have

$$\begin{array}{ccc} \mathrm{D}^b_{\mathrm{c}}(\mathcal{E}(\mathrm{D}),\Lambda) & \xrightarrow{\;c_!\;} & \mathrm{D}^b_{\mathrm{c}}(\mathcal{P}\mathrm{f}(\mathrm{D}),\Lambda) \\ {\scriptstyle p^*}\uparrow & & \\ \mathrm{D}^b_{\mathrm{c}}(\mathcal{P}\mathrm{f}(\mathrm{D})\times\mathcal{P}\mathrm{f}(\mathrm{D}),\Lambda) & & \end{array}$$

Now we set

$$\mu : \mathrm{D}^b_{\mathrm{c}}(\mathcal{P}\mathrm{f}(\mathrm{D})\times\mathcal{P}\mathrm{f}(\mathrm{D}),\Lambda) \longrightarrow \mathrm{D}^b_{\mathrm{c}}(\mathcal{P}\mathrm{f}(\mathrm{D}),\Lambda), \quad M \longmapsto c_! p^*(M)[\dim p]$$

Theorem (Theorem 10.3.5)**.** μ is associative.

We will also give a definition of *derived Hall category*, which will be the span of the *Lusztig sheaves* in the sense of [S2]. By restricting to the "abelian category part", we can reconstruct the Hall category in [S2], and have a geometric formulation of the ordinary Ringel-Hall algebra via the derived category of lisse-étale ℓ-adic constructible sheaves on the moduli stack of objects in the abelian category. In [S2, p.2], such a stacky construction is regarded as "(probably risky) project" and avoided. Thus our exposition will also give a new insight to the ordinal Ringel-Hall algebra.

We postpone the study of *the function-sheaf dictionary* for the lisse-étale constructible sheaves on derived stacks to a future article.

0.2. **Conventions and notations.** We will use some basic knowledge on

- dg-categories [T2, T4],
- model categories [H],
- simplicial homotopy theory [GJ],
- ∞-categories and ∞-topoi [Lur1],
- algebraic stacks in the ordinary sense [LM, O2]
- derived algebraic geometry in the sense of Toën-Vezzosi [TVe2, T6].

We give a brief summary of the theory of ∞-categories and ∞-topoi in §1, and explanations on some topics in Appendices B–D. Some recollections on algebraic spaces and algebraic stacks is given in Appendix A.

Here is a short list of our global conventions and notations.

- $\mathbb{N}$ denotes the set of non-negative integers.
- We fix two universes $\mathbb{U} \in \mathbb{V}$ and work on them. All the mathematical objects considered, such as sets, groups, rings and so on, are elements of $\mathbb{U}$ unless otherwise stated. For example, Set denotes the category of sets in $\mathbb{U}$. We will sometimes say an object is *small* if it belongs to $\mathbb{U}$.
- The word 'ring' means a unital and associative one unless otherwise stated, and the word '∞-category' means the one in [Lur1].
- A *poset* means a partially ordered set. A poset $(I, \le)$ is *filtered* if I is non-empty and for any $i, j \in I$ there exists $k \in I$ such that $i \le k$ and $j \le k$.
- A category will be identified with its nerve [Lur1, p.9] which is an ∞-category. Basically we denote an ordinary category in a serif font and denote an ∞-category in a sans-serif font. For example, in §2.2.2 we will denote by sCom the category of simplicial commutative rings, and by sCom the ∞-category of simplicial commutative rings.
- For categories C and D, the symbol $F : \mathrm{C} \rightleftarrows \mathrm{D} : G$ denotes an adjunction in the sense that we have an isomorphism $\mathrm{Hom}_{\mathrm{D}}(F(-), -) \simeq \mathrm{Hom}_{\mathrm{C}}(-, G(-))$ of functors. Thus it means $F \dashv G$. We use the same symbol $\rightleftarrows$ in the ∞-categorical context. See §B.5 for the detail.
- The homotopy category of a model category C will be denoted by $\mathrm{Ho}\,\mathrm{C}$, and the homotopy category of an ∞-category C will be denoted by $\mathrm{h}\,\mathsf{C}$. The functor category from a category C to another D will be denoted by $\mathrm{Fun}(\mathrm{C}, \mathrm{D})$.
- Finally, for an ∞-category C, the symbol $X \in \mathsf{C}$ means that X is an object of C.

Acknowledgements. The author is supported by the Grant-in-aid for Scientific Research (No. 16K17570, 19K03399), JSPS. This work is also supported by the JSPS Bilateral Program "Elliptic algebras, vertex operators and link invariant".

1. Notations on ∞-categories and ∞-topoi

In this subsection we explain some basic notions on ∞-categories and ∞-topoi which will be used throughout the main text. The purpose here is just to give brief accounts and introduce symbols of them. Somewhat detailed accounts of selected topics will also be given in Appendix B and Appendix C. All the contents are given in [Lur1, Lur2], and our terminology basically follow loc. cit., but we use slightly different symbols.

1.1. **Simplicial sets.** Let us start the explanation with our notations on simplicial sets. We follow [Lur1, §A.2.7] for symbols of simplicial sets.

Definition 1.1.1. We denote by $\mathbf{\Delta}$ the *category of combinatorial simplices*, which is described as

- An object is the linearly ordered set $[n] := \{0 \le 1 \le \cdots \le n\}$ for $n \in \mathbb{N}$.
- A morphisms is a non-strictly order-preserving function $[m] \to [n]$.

Definition. For a category C, a functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathrm{C}$ is called a *simplicial object in* C. In particular, a simplicial object in the category Set of sets is called a *simplicial set*.

We often use the symbol $S_n := S([n])$ for $n \in \mathbb{N}$. An element in S_0 is called a *vertex* of S, and an element in S_1 is called an *edge* of S. We write $v \in S$ to mean v is a vertex of S. A simplicial set S has the *face map* $d_j : S_n \to S_{n-1}$ and the *degeneracy map* $s_j : S_n \to S_{n+1}$ for each $j = 0, 1, \ldots, n$. See [Lur1, §A.2.7] for the precise definitions of d_j and s_j.

As noted in [Lur1, Remark A.2.7.1], the category $\mathbf{\Delta}$ is equivalent to the category $\mathrm{LinOrd}^{\mathrm{fin}}$ of all finite nonempty linearly ordered sets, and we sometimes identify them to regard simplicial sets (and more general simplicial objects) as functors which are defined on $\mathrm{LinOrd}^{\mathrm{fin}}$.

Using this convention, we introduce

Definition. For a simplicial set S, we define S^{op} to be the simplicial set given by

$$S^{\mathrm{op}}(J) := S(J^{\mathrm{op}}) \quad (J \in \mathrm{LinOrd}^{\mathrm{fin}}).$$

Here $J^{\mathrm{op}} \in \mathrm{LinOrd}^{\mathrm{fin}}$ has the same underlying set as J but with the opposite ordering of J. We call S^{op} the *opposite* of S.

Notation. The *category* Set_{Δ} *of simplicial sets* is defined to be the functor category

$$\mathrm{Set}_{\Delta} := \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Set}).$$

A morphism in Set_{Δ} is called a *simplicial map* or *a map of simplicial sets*.

The category Set_{Δ} has a model structure called the *Kan model structure*. See in Appendix B.1 for an account.

For a linearly ordered set J, we denote by Δ^J the simplicial set $[n] \mapsto \mathrm{Hom}([n], J)$, where the morphisms are taken in the category of linearly ordered sets. For $n \in \mathbb{N}$, we simply write $\Delta^n := \Delta^{[n]}$, and for $0 \le j \le n$, we denote by $\Lambda^n_j \subset \Delta^n$ the j-th horn [Lur1, Example A.2.7.3].

Finally we introduce

Definition. A *Kan complex* is a simplicial set K such that for any $n \in \mathbb{N}$ and any $0 \le i \le n$, any simplicial map $f_0 : \Lambda^n_i \to K$ admits an extension $f : \Delta^n \to K$.

We will use the following fact repeatedly in the main text.

Fact 1.1.2 ([Lur2, Corollary 1.3.2.12])**.** Any simplicial abelian group is a Kan complex.

1.2. **∞-categories.** Next we give some symbols for ∞-categories. We use the word "∞-category" in the sense of [Lur1].

Definition ([Lur1, Definition 1.1.2.4])**.** An ∞*-category* is a simplicial set C such that for any $n \in \mathbb{N}$ and any $0 < i < n$, any map $f_0 : \Lambda^n_i \to \mathsf{C}$ of simplicial sets admits an extension $f : \Delta^n \to \mathsf{C}$.

Since an ∞-category is a simplicial set, all the notions on simplicial sets can be transfered to those on an ∞-category. For example, we have

Definition. The *opposite* of an ∞-category C is defined to be the opposite C^{op} of C as a simplicial set in the sense of §1.1.

For an ∞-category C, the opposite C^{op} is an ∞-category. Thus we call C^{op} the *opposite* ∞*-category* of C.

Next we explain the relation between ordinary categories and ∞-categories.

Definition. For an ∞-category C, the *objects* are the vertices of C as a simplicial set, and the *morphisms* are the edges of C as a simplicial set.

Thus an object of an ∞-category C is a simplicial maps $\Delta^0 \to \mathsf{C}$. and a morphism of C is a simplicial maps $\Delta^1 \to \mathsf{C}$.

These notions are compatible with those in the ordinary category theory, which we now briefly explain.

Definition 1.2.1 ([Lur1, p.9])**.** The *nerve* of a category C, denoted by $\mathsf{N}(\mathrm{C})$, is a simplicial set with $\mathsf{N}(\mathrm{C})_n = \mathrm{Fun}([n], \mathrm{C})$, where the linearly ordered set $[n] = \{0, 1, \ldots, n\}$ is regarded as a category in the obvious way. The face maps and degeneracy maps are defined via compositions and inserting the identity morphisms.

By [Lur1, Proposition 1.1.2.2, Example 1.1.2.6], the nerve of any category C is an ∞-category. Then the objects and the morphisms of C coincide with those of $\mathsf{N}(\mathrm{C})$. Also we have $\mathsf{N}(\mathrm{C}^{\mathrm{op}}) = \mathsf{N}(\mathrm{C})^{\mathrm{op}}$ as simplicial sets.

Next we introduce functors of ∞-categories. For that, let us recall

Definition 1.2.2 ([GJ, Chap. 1 §5])**.** For $X, Y \in \mathrm{Set}_\Delta$, we define the simplicial set $\mathrm{Map}_{\mathrm{Set}_\Delta}(X, Y)$ by the following description.

- $\mathrm{Map}_{\mathrm{Set}_\Delta}(X, Y)_n := \mathrm{Hom}_{\mathrm{Set}_\Delta}(X \times \Delta^n, Y)$ for each $n \in \mathbb{N}$, where $\times$ denotes the product of simplicial sets.
- The degeneracy maps and the face maps are induced by those on Δ^n.

The simplicial set $\mathrm{Map}_{\mathrm{Set}_\Delta}(X, Y)$ is called the function complex in the literature of the simplicial homotopy theory, but we will not use this word.

Definition 1.2.3 ([Lur1, Notation 1.2.7.2])**.** Let K be a simplicial set and C be an ∞-category. We set

$$\mathsf{Fun}(K, \mathsf{C}) := \mathrm{Map}_{\mathrm{Set}_\Delta}(K, \mathsf{C}).$$

If $K = \mathsf{B}$ is an ∞-category, then $\mathsf{Fun}(\mathsf{B}, \mathsf{C})$ is called the ∞-*category of functors* from B to C, and its object, i.e. a simplicial map $\mathsf{B} \to \mathsf{C}$, is called a *functor*.

Thus a functor of ∞-category is nothing but a simplicial map. The simplicial set $\mathsf{Fun}(K, \mathsf{C})$ is an ∞-category by [Lur1, Proposition 1.2.7.3 (1)].

1.3. **The homotopy category of an ∞-category.** The definition of the homotopy category of an ∞-category is rather complicated, so we postpone a detailed explanation to Appendix §B.2, and here we only give relevant important notions.

For the definition of the homotopy category of an ∞-category, we need to recall the notion of a topological category. We will use the terminology on enriched categories (see [Lur1, §A.1.4] for example).

Definition 1.3.1 ([Lur1, Definition 1.1.1.6])**.**
(1) We denote by $\mathcal{CG}$ the category of compactly generated weakly Hausdorff topological spaces.
(2) A *topological category* is defined to be a category enriched over $\mathcal{CG}$.
(3) For a topological category $\mathcal{C}$ and its objects $X, Y \in \mathcal{C}$, we denote by $\mathrm{Map}_{\mathcal{C}}(X, Y) \in \mathcal{CG}$ the topological space of morphisms and call it the *mapping space*.

Next we introduce the category $\mathcal{H}$ called the *homotopy category of spaces*. Let $\mathcal{CW}$ be the topological category whose objects are CW complexes and $\mathrm{Map}_{\mathcal{CW}}(X, Y)$ is the set of continuous maps equipped with the compact-open topology.

Definition 1.3.2 ([Lur1, Example 1.1.3.3])**.** The *homotopy category $\mathcal{H}$ of spaces.* is the category defined as follows.

- The objects of $\mathcal{H}$ are defined to be the objects of $\mathcal{CW}$.
- For $X, Y \in \mathcal{C}$, we set $\mathrm{Hom}_{\mathcal{H}}(X, Y) := \pi_0(\mathrm{Map}_{\mathcal{CW}}(X, Y))$.
- Composition of morphisms in $\mathcal{H}$ is given by the application of π_0 to composition of morphisms in $\mathcal{C}$.

The category $\mathcal{H}$ is actually the homotopy category of $\mathcal{CW}$ in the sense of [Lur1, Definition 1.1.3.2].

Next we recall

Definition 1.3.3. A category enriched in Set_Δ is called a *simplicial category.* We denote by Cat_Δ the category of simplicial categories. A Set_Δ-enriched functor between simplicial categories is called a *simplicial functor.*

The category Set_Δ itself is a simplicial category by the simplicial set $\mathrm{Map}_{\mathrm{Set}_\Delta}(\cdot, \cdot)$ in Definition 1.2.2. Then there is a functor

$$\mathfrak{C}[\cdot] : \mathrm{Set}_\Delta \longrightarrow \mathrm{Cat}_\Delta \tag{1.3.1}$$

by [Lur1, §1.1.5] (see also Appendix B.2). Then the *homotopy category* $\mathrm{h}\,S$ of a simplicial set S is defined to be

$$\mathrm{h}\,S := \mathrm{h}\,\mathfrak{C}[S],$$

where the right hand side denotes the homotopy category of the simplicial category $\mathfrak{C}[S]$ (Definition B.2.2).

Definition. The *homotopy category* of an ∞-category C is defined to be the homotopy category $\mathrm{h}\,\mathsf{C}$ of C as a simplicial set.

For a simplicial set S, the homotopy category $\mathrm{h}\,S$ is enriched over $\mathcal{H}$. Thus, for vertices $x, y \in S$ we may denote

$$\mathrm{Map}_S(x, y) := \mathrm{Hom}_{\mathrm{h}\,S}(x, y) \in \mathcal{H}.$$

It is called the *mapping space* from x to y in S.

We see that a simplicial map $f : S \to S'$ induces a functor $\mathrm{h}\,f : \mathrm{h}\,S \to \mathrm{h}\,S'$ between the homotopy categories. Thus the following definitions make sense.

Definition 1.3.4 ([Lur1, Definition 1.2.10.1])**.** Let $f : S \to S'$ be a simplicial map.

(1) f is called *essentially surjective* if the induced functor $\mathrm{h}\,f$ is essentially surjective.
(2) f is called *fully faithful* if $\mathrm{h}\,f$ is a fully faithful functor of $\mathcal{H}$-enriched categories.
(3) f is called an *equivalence* if it is essentially surjective and fully faithful.

For a morphism g of ∞-categories, the same terminology will be used with g regarded as a simplicial map.

These definitions are compatible with the notions of the ordinary category theory. In fact, for a category C and its objects $X, Y \in \mathrm{C}$, we have a bijection $\mathrm{Hom}_{\mathrm{C}}(X, Y) = \pi_0 \mathrm{Map}_{\mathsf{N}(\mathrm{C})}(X, Y)$, where in the right hand side we regard X and Y as objects of the ∞-category $\mathsf{N}(\mathrm{C})$.

For later use, we also introduce

Definition 1.3.5. A functor $f : \mathsf{B} \to \mathsf{C}$ of ∞-categories is *conservative* if the following condition is satisfied: if β is a morphism in B such that $f(\beta)$ is an equivalence in C, then β is an equivalence in B.

Definition 1.3.6 ([Lur1, §1.2.11])**.** Let C be an ∞-category.

(1) For a subcategory $\mathrm{B} \subset \mathrm{h}\,\mathsf{C}$, define B to be the simplicial set appearing in the following pullback diagram of simplicial sets (see [GJ, Chap. 1] for the pullback in Set_Δ).

$$\begin{array}{ccc} \mathsf{B} & \longrightarrow & \mathsf{C} \\ \downarrow & & \downarrow \\ \mathsf{N}(\mathrm{B}) & \longrightarrow & \mathsf{N}(\mathrm{h}\,\mathsf{C}) \end{array}$$

Then B is an ∞-category and called the *sub-∞-category of* C *spanned by* B.
(2) A simplicial subset $\mathsf{C}' \subset \mathsf{C}$ is a *sub-∞-category of* C if it arises by this construction.
(3) If in the item (1) the subcategory $\mathrm{B} \subset \mathrm{h}\,\mathsf{C}$ is a full subcategory, then the subcategory B is called the *full sub-∞-category of* C *spanned by* B. Similarly, a simplicial subset $\mathsf{C}' \subset \mathsf{C}$ arising in this way is called a *full sub-∞-category*.

We close this part with

Definition 1.3.7 ([Lur1, Definition 1.2.12.1])**.** Let C be an ∞-category.

(1) An object $\mathbf{1}_{\mathsf{C}} \in \mathsf{C}$ is called a *final object* if for any $X \in \mathsf{C}$ the mapping space $\mathrm{Map}_{\mathsf{C}}(X, \mathbf{1}_{\mathsf{C}})$ is a final object in $\mathcal{H}$, i.e., a weakly contractible space.
(2) An object $\emptyset \in \mathsf{C}$ is called an *initial object* if it is a final object of C^{op}.

The following is a characterization of final objects in an ∞-category.

Fact 1.3.8 ([Lur1, Definition 1.2.12.3, Proposition 1.2.12.4, Corollary 1.2.12.5])**.** For an object X of an ∞-category C, the followings are equivalent.

- X is a final object.
- The canonical functor $\mathsf{C}_{/X} \to \mathsf{C}$ (Corollary B.3.2) is a trivial fibration of simplicial sets with respect to the Kan model structure (Fact B.1.2).
- The Kan complex $\mathrm{Hom}^R_{\mathsf{C}}(Y, X)$ (Definition B.2.5) representing the mapping space $\mathrm{Map}_{\mathsf{C}}(Y, X)$ is contractible for any $Y \in X$.

For later use, let us cite

Fact 1.3.9 ([Lur1, Proposition 1.2.12.9])**.** The sub-∞-category spanned by final objects in an ∞-category either is empty or is a contractible Kan complex.

1.4. **Simplicial nerves and the ∞-category of spaces.** Let us introduce the ∞-category of spaces in the sense of [Lur1]. We begin with

Definition 1.4.1 ([Lur1, Definition 1.1.5.6])**.** For a simplicial category $\mathfrak{C}$, there is a simplicial set $\mathsf{N}_{\mathrm{spl}}(\mathfrak{C})$ characterized by the property

$$\mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^n, \mathsf{N}_{\mathrm{spl}}(\mathfrak{C})) = \mathrm{Hom}_{\mathrm{Cat}_\Delta}(\mathfrak{C}[\Delta^n], \mathfrak{C}).$$

It is called the *simplicial nerve* of $\mathfrak{C}$.

We have the following statements for simplicial nerves.

Fact 1.4.2 ([Lur1, Proposition 1.1.5.10, Remark 1.2.16.2])**.**

(1) For a simplicial category $\mathfrak{C}$, the simplicial nerve $\mathsf{N}_{\mathrm{spl}}(\mathfrak{C})$ is an ∞-category such that the simplicial set $\mathrm{Map}_{\mathfrak{C}}(X,Y)$ is a Kan complex for every $X, Y \in \mathfrak{C}$.

(2) For any $X, Y \in \mathcal{K}\mathrm{an}$, the simplicial set $\mathrm{Map}_{\mathcal{K}\mathrm{an}}(X,Y)$ is a Kan complex.

By this fact, the following definition makes sense.

Definition 1.4.3 ([Lur1, Definition 1.2.16.1])**.** Let $\mathcal{K}\mathrm{an}$ be the full subcategory of Set_Δ spanned by the collection of small Kan complexes, considered as a simplicial category via $\mathrm{Map}_{\mathcal{K}\mathrm{an}}(-,-) \subset \mathrm{Map}_{\mathrm{Set}_\Delta}(-,-)$. We set

$$\mathcal{S} := \mathsf{N}_{\mathrm{spl}}(\mathcal{K}\mathrm{an})$$

and call it $\mathcal{S}$ *the ∞-category of spaces*.

Remark ([Lur1, Example 1.1.5.8])**.** An ordinary category C can be regarded as a simplicial category by setting each simplicial set $\mathrm{Hom}_{\mathrm{C}}(X,Y)$ to be constant. Then the simplicial nerve $\mathsf{N}_{\mathrm{spl}}(\mathrm{C})$ of this simplicial category C agrees with the nerve $\mathsf{N}(\mathrm{C})$ of C as an ordinary category (Definition 1.2.1).

Let us recall a classic result on $\mathcal{S}$ due to Quillen, which justifies the name "homotopy category of spaces" of $\mathcal{H}$.

Fact (Quillen)**.** As for the homotopy category we have

$$\mathrm{h}\,\mathcal{S} \simeq \mathcal{H}.$$

This equivalence is induced by the adjunction $|-| : \mathrm{Set}_\Delta \rightleftarrows \mathcal{CG} : \mathrm{Sing}$ in (B.1.1). See [GJ, Chap. 1, Theorem 11.4] for a proof.

1.5. **Presheaves and ∞-categorical Yoneda embedding.** Here we cite from [Lur1, §5.1] an analogue of the Yoneda embedding in the theory of ∞-categories. We begin with

Definition 1.5.1 ([Lur1, Definition 5.1.0.1])**.** For a simplicial set K, we define

$$\mathsf{PSh}(K) := \mathsf{Fun}(K^{\mathrm{op}}, \mathcal{S})$$

and call it the ∞-category of *presheaves of spaces* on K. Its object is called a *presheaf of spaces* on K.

The following is a fundamental property of $\mathsf{PSh}(K)$. For the notion of (co)limits in ∞-categories, see Appendix B.4.

Fact. For any simplicial set K, the ∞-category $\mathsf{PSh}(K)$ is a presentable (Definition B.8.3). In particular, it admits small limits and small colimits (Fact B.8.5).

For a simplicial set K, let $\mathfrak{C}[K]$ be the associated simplicial category given at (1.3.1) in §1.3. Recall also the simplicial category $\mathcal{K}\mathrm{an}$ of Kan complexes (Definition 1.4.3). Then we can consider the following simplicial functor (Definition 1.3.3):

$$\mathfrak{C}[K]^{\mathrm{op}} \times \mathfrak{C}[K] \longrightarrow \mathcal{K}\mathrm{an}, \quad (X,Y) \longmapsto \mathrm{Sing}\left|\mathrm{Hom}_{\mathfrak{C}[K]}(X,Y)\right|.$$

Here we used the adjunction (B.1.1). We also have a natural simplicial functor $\mathfrak{C}[K^{\mathrm{op}} \times K] \to \mathfrak{C}[K]^{\mathrm{op}} \times \mathfrak{C}[K]$. Composing these functors, we have a simplicial functor

$$\mathfrak{C}[K^{\mathrm{op}} \times K] \longrightarrow \mathcal{K}\mathrm{an}.$$

Then recalling Definition 1.4.1 of the simplicial nerve and Definition 1.4.3 of $\mathcal{S}$, by passing to the adjoint we have a simplicial map $K^{\mathrm{op}} \times K \to \mathcal{S}$ of simplicial sets. It gives rise to a simplicial map

$$j : K \longrightarrow \mathsf{PSh}(K) = \mathsf{Fun}(K^{\mathrm{op}}, \mathcal{S}).$$

Then by [Lur1, Proposition 5.1.3.1] j is fully faithful (Definition 1.3.4).

Definition 1.5.2. For a simplicial set K, the fully faithful simplicial map $j : K \to \mathsf{PSh}(K)$ is called the *Yoneda embedding* of K.

See [TVe1, §2.4] for an account of the Yoneda embedding in the model categorical setting.
For the later use, we cite

Fact 1.5.3 ([Lur1, Corollary 5.1.5.8])**.** For an ∞-category C, the ∞-category $\mathsf{PSh}(\mathsf{C})$ is freely generated under small colimits by the image of the Yoneda embedding C.

1.6. **The ∞-category of small ∞-categories.** For later use, let us now introduce

Definition 1.6.1 ([Lur1, Definition 3.0.0.1])**.**
(1) Define the simplicial category $\mathfrak{Cat}_\infty$ as follows.
- The objects are small ∞-categories.
- For ∞-categories B and C, define $\mathrm{Map}_{\mathfrak{Cat}_\infty}(\mathsf{B},\mathsf{C})$ to be the largest Kan complex contained in the ∞-category $\mathsf{Fun}(\mathsf{B},\mathsf{C})$.

(2) We set
$$\mathsf{Cat}_\infty := \mathsf{N}_{\mathrm{spl}}(\mathfrak{Cat}_\infty)$$
and call it the *∞-category of small ∞-categories*.

Note that Cat_∞ is indeed an ∞-category by Fact 1.4.2 (1). Furthermore, by [Lur1, §3.3.3, §3.3.4, Corollary 4.2.4.8], Cat_∞ admits small limits and small colimits.

Applying various constructions for simplicial sets to Cat_∞, we obtain the corresponding constructions for ∞-categories. For example, we have

Definition 1.6.2. For functors $f : \mathsf{B} \to \mathsf{D}$ and $g : \mathsf{C} \to \mathsf{D}$ of small ∞-categories, we have the *fiber product* $\mathsf{B} \times_{f,\mathsf{D},g} \mathsf{C}$ by applying Definition B.4.5 of fiber product to the ∞-category Cat_∞.

1.7. **∞-sites.** In this subsection we recall the notion of Grothendieck topology on an ∞-category and a construction of ∞-topos following [Lur1, §6.2.2]. See also [TVe1] for a presentation in the theory of model category.

Using the notion of over-∞-category (Appendix §B.3), we introduce

Definition ([Lur1, Definition 6.2.2.1])**.** Let C be an ∞-category.
(1) A *sieve* on C is a full sub-∞-category $\mathsf{C}^{(0)} \subset \mathsf{C}$ such that if $f : X \to Y$ is a morphism in C and $Y \in \mathsf{C}^{(0)}$, then X also belongs to $\mathsf{C}^{(0)}$.
(2) For $X \in \mathsf{C}$, a *sieve on X* is a sieve on the over-∞-category $\mathsf{C}_{/X}$.

Next we want to introduce the pullback of sieves. For that, we prepare

Lemma 1.7.1. Given a functor $F : \mathsf{B} \to \mathsf{C}$ of ∞-categories and a sieve $\mathsf{C}^{(0)} \subset \mathsf{C}$,
$$F^{-1}\mathsf{C}^{(0)} := \mathsf{C}^{(0)} \times_{\mathsf{C}} \mathsf{B} \subset \mathsf{B}$$
is a sieve on B. Here the right hand side denotes the fiber product of ∞-categories (Definition 1.6.2).

The proof is obvious.

Recall also that by Corollary B.3.2, given a morphism $f : X \to Y$ in C, we have a morphism $f_* : \mathsf{C}_{/X} \to \mathsf{C}_{/Y}$ of over-∞-categories. Applying Lemma 1.7.1 to $F = f_*$, we have

Definition. Let C and $f : X \to Y$ as above. Given a sieve $\mathsf{C}^{(0)}_{/Y}$ on Y, we define the *pullback* $f^*\mathsf{C}^{(0)}_{/Y}$ to be $f^*\mathsf{C}^{(0)}_{/Y} := (f^*)^{-1}\mathsf{C}^{(0)}_{/Y}$.

Definition 1.7.2 ([Lur1, Definition 6.2.2.1])**.** Let C be an ∞-category.
(1) A *Grothendieck topology* τ on C consists of a collection $\mathrm{Cov}(X)$ of sieves on each $X \in \mathsf{C}$, called *covering sieves* on X, satisfying the following conditions.
  (a) For each $X \in \mathsf{C}$, the over-∞-category $\mathsf{C}_{/X}$ as a sieve on X belongs to $\mathrm{Cov}(X)$.
  (b) For each morphism $f : X \to Y$ in C and each $\mathsf{C}^{(0)}_{/Y} \in \mathrm{Cov}(Y)$, the pullback $f^*\mathsf{C}^{(0)}_{/Y}$ belongs to $\mathrm{Cov}(X)$.
  (c) Let $Y \in \mathsf{C}$ and $\mathsf{C}^{(0)}_{/Y} \in \mathrm{Cov}(Y)$. If $\mathsf{C}^{(1)}_{/Y}$ be a sieve on Y such that $f^*\mathsf{C}^{(1)}_{/Y} \in \mathrm{Cov}(X)$ for any $f : X \to Y$ in $\mathsf{C}^{(0)}_{/Y}$, then $\mathsf{C}^{(1)}_{/Y} \in \mathrm{Cov}(Y)$.

We denote $\mathrm{Cov}_\tau(X) := \mathrm{Cov}(X)$ to emphasize that it is associated to τ.
(2) A pair (C, τ) of an ∞-category C and a Grothendieck topology τ on C is called an *∞-site*.

Remark. As mentioned in [Lur1, Remark 6.2.2.3], if C is the nerve of an ordinary category, then the above notion reduces to the ordinary definitions of a sieve and a Grothendieck topology [SGA4, II §1]. Indeed, as for the definition of a Grothendieck topology, we have

- The condition (a) in Definition 1.7.2 (1) implies that the one-member family $\{\mathrm{id} : X \to X\}$ is a covering sieve on X (the condition T3 in loc. cit.).
- The condition (b) implies that collections of covering sieves are stable under fiber product (the condition T1 in loc. cit.).
- The condition (c) is nothing but the local character (the condition T2 in loc. cit.).

Remark 1.7.3. In [Lur1, Remark 6.2.2.3] it is explained that giving a Grothendieck topology on an ∞-category C is equivalent to giving a Grothendieck topology in the ordinary sense on the homotopy category $\mathrm{h}\,\mathsf{C}$. The latter one is equivalent to the definition of a Grothendieck topology on C in [TVe1, TVe2, T6].

In the main text we construct a Grothendieck topology on an ∞-category using this equivalence. Thus we will only specify the data of covering sieves on the homotopy category of the given ∞-category.

Having introduced the notion of ∞-sites, we now define *sheaves on an ∞-site*.

Let (C, τ) be an ∞-site. One can construct a set S_τ of monomorphisms $U \to j(X)$ corresponding to covering sieves of $X \in \mathsf{C}$ of τ (Definition C.1.2). Then a presheaf $\mathcal{F} \in \mathsf{PSh}(\mathsf{C})$ is called a *τ-sheaf* if it is S_τ-local. We denote by

$$\mathsf{Sh}(\mathsf{C}, \tau) \subset \mathsf{PSh}(\mathsf{C})$$

the full sub-∞-category spanned by τ-sheaves.

The ∞-category $\mathsf{Sh}(\mathsf{C}, \tau)$ has the following properties.

Fact 1.7.4 ([Lur1, Proposition 6.2.2.7])**.** Let (C, τ) be an ∞-site.

(1) The ∞-category $\mathsf{Sh}(\mathsf{C}, \tau)$ is a *topological localization* [Lur1, §6.2.1] of $\mathsf{PSh}(\mathsf{C})$. We call the localization functor $\mathsf{PSh}(\mathsf{C}) \to \mathsf{Sh}(\mathsf{C}, \tau)$ the *sheafification functor*.
(2) The ∞-category $\mathsf{Sh}(\mathsf{C}, \tau)$ is an ∞-topos. We call it the *associated ∞-topos* of (C, τ).

The definition of an ∞-topos will be given in the next subsection.

1.8. **∞-topoi.** Recall the notion of an accessible functor (Definition B.9.3), a left exact functor (Definition B.10.1) and a localization functor (Definition B.7.2).

Definition 1.8.1 ([Lur1, Definition 6.1.0.4])**.** An ∞-category T is called an *∞-topos* if there exist a small ∞-category B and an accessible left exact localization functor $\mathsf{PSh}(\mathsf{B}) \to \mathsf{T}$.

This definition makes sense since $\mathsf{PSh}(\mathsf{B})$ is accessible for any small ∞-category B by [Lur1, Example 5.4.2.7, Proposition 5.3.5.12] so that we can ask if a functor from $\mathsf{PSh}(\mathsf{B})$ is accessible or not.

Fact 1.7.4 can now be shown by taking $\mathsf{T} = \mathsf{Sh}(\mathsf{C}, \tau)$ and $\mathsf{B} = \mathsf{C}$ since we have $\mathsf{Sh}(\mathsf{C}, \tau) = \mathsf{PSh}(\mathsf{C})[S_\tau^{-1}]$.

The next statement is obvious from the definitions.

Lemma 1.8.2. Let $\mathsf{T} := \mathsf{Sh}(\mathsf{C}, \tau)$ be the ∞-topos obtained from an ∞-site (C, τ). Taking π_0 in the value of objects in T, one gets an topos in the ordinary sense [SGA4] on the underlying category of C. We denote the obtained topos by T^{cl}.

As for a general ∞-topos, we have the following Giraud-type theorem.

Fact 1.8.3 ([Lur1, Theorem 6.1.0.6])**.** For an ∞-category T, the following conditions are equivalent.

(i) T is an ∞-topos.
(ii) T satisfies the following four conditions: (a) T is presentable. (b) Colimits in T are universal [Lur1, Definition 6.1.1.2]. (c) Coproducts in T are disjoint [Lur1, §6.1.1, p.532]. (d) Every groupoid object of T is effective [Lur1, §6.1.2].

We will repeatedly use the following consequences.

Corollary 1.8.4. An ∞-topos admits arbitrary small limits and small colimits.

Proof. Since an ∞-topos is presentable by Fact 1.8.3, it admits small colimits by Definition B.8.3, and admits small limits by Fact B.8.5 (1). □

Corollary 1.8.5. An ∞-topos T has an initial object, which will be typically denoted by $\emptyset_{\mathsf{T}}$. Also T has a final object, typically denoted by $\mathbf{1}_{\mathsf{T}}$.

Proof. An initial object is a colimit of the empty diagram, so that $\emptyset_{\mathsf{T}}$ exists by Corollary 1.8.4. Similarly a final object exists since it is a limit of the empty diagram. □

Let us also cite

Fact 1.8.6 ([Lur1, Proposition 6.3.5.1 (1)])**.** For an ∞-topos T and an object $U \in \mathsf{T}$, the over-∞-category $\mathsf{T}_{/U}$ is an ∞-topos. We call it the *localized ∞-topos* of T on U, and sometimes denote it by $\mathsf{T}|_U$.

See Fact 3.1.5 for a complementary explanation of the localized ∞-topos.
Let us now introduce another notion of sheaves.

Definition 1.8.7 ([Lur1, Notation 6.3.5.16])**.** For ∞-topoi T and C admitting small limits, we denote the full sub-∞-category of $\mathsf{Fun}(\mathsf{T}^{\mathrm{op}}, \mathsf{C})$ spanned by functors preserving small limits by

$$\mathsf{Shv}_{\mathsf{C}}(\mathsf{T}) \subset \mathsf{Fun}(\mathsf{T}^{\mathrm{op}}, \mathsf{C}),$$

and call it the *∞-category of C-valued sheaves on* T. We usually assume T to be an ∞-topos.

This notion of sheaves on an ∞-topos will be fundamental for our study of sheaves on derived stacks (§4, §5). At first it looks strange since the definition does not include the standard sheaf conditions. However, as the following fact implies, it behaves nicely on an ∞-topos.

Fact 1.8.8 ([Lur1, Remark 6.3.5.17])**.** Let T be an ∞-topos. Then the Yoneda embedding $\mathsf{T} \to \mathsf{Shv}_{\mathcal{S}}(\mathsf{T})$ is an equivalence.

See Appendix §C.2 for account on the ∞-categorical Yoneda embedding.
Another remark on Shv is that we have the following compatibility with the symbol Sh (Definition C.1.2).

Fact ([Lur5, Proposition 1.1.12])**.** Let (C, τ) be an ∞-site with C admitting small limits, and B be an ∞-category admitting small limits. Then we have the following equivalence of ∞-categories.

$$\mathsf{Shv}_{\mathsf{B}}(\mathsf{Sh}(\mathsf{C}, \tau)) \xrightarrow{\sim} \mathsf{Shv}_{\mathsf{B}}(\mathsf{C}).$$

For later use, we introduce

Definition 1.8.9. Let T be an ∞-topos and $S \in \mathcal{S}$. The *constant sheaf* valued in S on T is an object of $\mathsf{Shv}_{\mathcal{S}}(\mathsf{T})$ determined by the correspondence $U \mapsto S$ for any $U \in \mathsf{T}$. It will be denoted by the same symbol S.

We close this subsection by introducing basic notions on ∞-topoi.

Definition 1.8.10 ([Lur1, Corollary 6.2.3.5])**.** A morphism $f : U \to V$ in an ∞-topos T is called an *effective epimorphism* if as an object $f \in \mathsf{T}_{/V}$ the truncation $\tau_{\le -1}(f)$ is a final object (Definition 1.3.7) of $\mathsf{T}_{/V}$.

Here we used the truncation functor $\tau_{\le -1} : \mathsf{T}_{/V} \to \tau_{\le -1}\mathsf{T}_{/V}$ in Definition B.9.4. This definition makes sense since $\mathsf{T}_{/V}$ is an ∞-topos by Fact 1.8.6 so that it is presentable by Fact 1.8.3.
Using the notion of effective epimorphism, we can introduce analogues of ordinary notions of topoi. We also use *coproducts* in an ∞-category (Definition B.4.4).

Definition 1.8.11. A *covering* of an ∞-topos T is an effective epimorphism $\coprod_{i \in I} U_i \to \mathbf{1}_{\mathsf{T}}$, where T denotes a final object of T (Corollary 1.8.5). We denote it by $\{U_i\}_{i \in I}$, suppressing the morphism to T, if no confusion will occur.

We have an obvious notion of *subcovering* of a covering of an ∞-topos.

Definition 1.8.12 ([Lur7, Definition 3.1])**.** Let T be an ∞-topos.
(1) An ∞-topos T is *quasi-compact* if any covering of T has a finite subcovering.
(2) An object $T \in \mathsf{T}$ is *quasi-compact* if the ∞-topos $\mathsf{T}_{/T}$ is quasi-compact in the sense of (1).

2. Recollections on derived stacks

In this section we explain the theory of derived stacks which will be used throughout the main text. The main references are [TVe2, T6].

Our presentation is based on the ∞-categorical language, although the fundamentals of the theory of derived stacks is developed in [TVe1, TVe2] via the model theoretic language. We refer [Lur1] for the ∞-categorical language used here.

2.1. **Higher Artin stacks.** We cite from [TVe2, §2.1] the notion of higher Artin stacks, which enables one to develop an extension of the ordinary theory of algebraic stacks. Although higher Artin stacks will not play essential roles in our study, we give a summary of the theory by the following two reasons.

- The theory of higher Artin stacks and the theory of derived stacks are developed in a parallel way in [TVe2, §2.1, §2,2], and in the case of higher Artin stacks some parts of the theory are simple. We give an explanation on higher Artin stacks as a warming-up for the theory of derived stacks. The latter one will be explained in the next §2.2.
- Our discussion on derived Hall algebra has a non-derived counterpart which is developed in the region of ordinary Artin stacks. The theory of ordinary Artin stack is naturally embedded in the theory of higher Artin stacks, so we need some terminology on higher Artin stacks.

2.1.1. *Definition.* Fix a commutative ring k.

Let us consider the (ordinary) category Com_k of commutative k-algebras. We denote the ∞-category of the nerve of Com_k by

$$\mathsf{Com}_k := \mathsf{N}(\mathrm{Com}_k).$$

Remark. Let us explain other ways to define Com_k, all of which give equivalent ∞-categories in the sense of Definition 1.3.4.

- One way is to use the ∞-localization (Definition B.7.1) to the pair (Com_k, W), where W is the set of ring isomorphisms.
- Another way is to use the simplicial nerve (Definition 1.4.1). Let $\mathfrak{Com}_k$ be the simplicial category (Definition 1.3.3) of commutative k-algebras with the simplicial set $\mathrm{Map}_{\mathfrak{Com}_k}(X,Y)$ set to be $\mathrm{Hom}_{\mathrm{Com}_k}(X,Y)$ regarded as a constant simplicial set. Then we have an ∞-category $\mathsf{N}_{\mathrm{spl}}(\mathfrak{Com}_k)$.

Definition. (1) Com_k is called the *∞-category of commutative k-algebras.*

(2) The *∞-category Aff_k of affine schemes* is defined to be the opposite ∞-category of Com_k:

$$\mathsf{Aff}_k := (\mathsf{Com}_k)^{\mathrm{op}}.$$

The object of Aff_k corresponding to $A \in \mathsf{Com}_k$ will be denoted by $\operatorname{Spec} A$ and called the *affine scheme* of A.

(3) In the case $k = \mathbb{Z}$, we sometimes suppress the subscript and denote $\mathsf{Com} := \mathsf{Com}_{\mathbb{Z}}$ and $\mathsf{Aff} := \mathsf{Aff}_{\mathbb{Z}}$.

This definition is just an analogue of the ordinary scheme theory: the category Aff of affine schemes is equivalent to the opposite category $\mathrm{Com}^{\mathrm{op}}$ of the category of commutative rings.

The ∞-category Com_k can be seen as the category Com_k of commutative k-algebras equipped with "higher structure" on the set of morphisms. See §B for a summary of the theory of ∞-categories. In particular, a morphism in the category Aff_k of affine schemes can be regarded as a morphism of the ∞-category Aff_k. Thus ordinary notions on morphisms of affine schemes can be transfered to those in Aff_k.

Let us now consider the ∞-category

$$\mathsf{PSh}(\mathsf{Aff}_k) := \mathsf{Fun}\left((\mathsf{Aff}_k)^{\mathrm{op}}, \mathcal{S}\right)$$

of presheaves of spaces over Aff_k (Definition 1.5.1). An object of $\mathsf{PSh}(\mathsf{Aff}_k)$ is a functor $\mathsf{Aff}_k \to \mathcal{S}$ of ∞-categories, where $\mathcal{S}$ denotes the ∞-category of spaces (Definition 1.4.3).

Remark. A few ethical remarks are in order.

(1) In Lurie's theory of derived algebraic geometry [Lur5]–[Lur14], the ∞-category $\mathcal{S}$ of spaces is considered to be a correct ∞-theoretical replacement of the ordinary category Set of sets.

(2) In [TVe1, TVe2], a functor to the model category Set_Δ of simplicial sets is called a prestack. Since all the prestacks appearing in our discussion are valued in Kan complexes, we replace Set_Δ by $\mathcal{S}$.

Next we consider a Grothendieck topology on the ∞-category Aff_k. See Appendix C for the relevant notions.

By [TVe2, Lemma 2.1.1.1], étale coverings of affine schemes give a (non-∞-theoretical) Grothendieck topology on the homotopy category $\mathrm{h}\,\mathsf{Aff}_k$. Then by Remark 1.7.3 we have a Grothendieck topology on the ∞-category Aff_k.

Definition. The obtained Grothendieck topology on Aff_k is denoted by et and called the *étale topology* on Aff_k.

Now we apply the construction of an ∞-topos in §1.8 to the ∞-site $(\mathsf{Aff}_k, \mathrm{et})$. We consider the ∞-category

$$\mathsf{Sh}(\mathsf{Aff}_k, \mathrm{et})$$

of et-sheaves on Aff_k (see §1.7 for an account, and Definition C.1.2 for a strict definition). By Fact 1.7.4, the ∞-category $\mathsf{Sh}(\mathsf{Aff}_k, \mathrm{et})$ is an ∞-topos (Definition 1.8.1), which is a localization of the ∞-category $\mathsf{PSh}(\mathsf{Aff}_k)$. Not that by Corollary C.3.5, this ∞-topos $\mathsf{Sh}(\mathsf{Aff}_k, \mathrm{et})$ is hypercomplete (Definition C.3.1).

Definition 2.1.1 ([TVe2, Definition 1.3.2.2, 1.3.6.3])**.** The *∞-category* St_k *of stacks over* k is defined to be

$$\mathsf{St}_k := \mathsf{Sh}(\mathsf{Aff}_k, \mathrm{et}).$$

An object in St_k is called a *stack* (*over* k), and a morphism in St_k is called a *morphism of stacks* (*over* k).

Remark 2.1.2. Let us remark that in [TVe2] the word "stack" is used in a slightly different way. Denoting by $\mathrm{h}\,\mathsf{C}$ the homotopy category of an ∞-category C (Definition B.2.3), we have an adjunction

$$a : \mathrm{h}\,\mathsf{PSh}(\mathsf{Aff}_k) \rightleftarrows \mathrm{h}\,\mathsf{Sh}(\mathsf{Aff}_k, \mathrm{et}) : j$$

between the homotopy categories of $\mathsf{Sh}(\mathsf{Aff}, \mathrm{et})$ and $\mathsf{PSh}(\mathsf{Aff})$ with j fully faithful. In [TVe2] a stack means an object in $\mathsf{PSh}(\mathsf{Aff}_k)$ whose class in the homotopy category $\mathrm{h}\,\mathsf{PSh}(\mathsf{Aff}_k)$ is in the essential image of the functor j. This terminology and Definition 2.1.1 are different, but we can identify them up to a choice of equivalence in St_k. Similarly, a morphism of stacks is defined in [TVe2] to be a morphism in $\mathrm{h}\,\mathsf{PSh}(\mathsf{Aff}_k)$, which differs from our definition. Since there is no essential difference in our study, we will use Definition 2.1.1 only.

Recall the definition of the set $\pi_0(S)$ of path components of a simplicial set S (see [GJ, §1.7] for example). Since a stack is a sheaf valued in $\mathcal{S}$ and so is a sheaf of simplicial sets, the following notation makes sense.

Definition.
(1) For a stack $\mathcal{X} \in \mathsf{St}_k$, we denote by $\pi_0(\mathcal{X}) \in \mathrm{Fun}((\mathsf{Aff}_k)^{\mathrm{op}}, \mathrm{Set})$ the sheaf of sets obtained by taking π_0.
(2) For a morphism $f : \mathcal{X} \to \mathcal{Y}$ of stacks, we denote by $\pi_0(f)$ the induced morphism $\pi_0(\mathcal{X}) \to \pi_0(\mathcal{Y})$ of sheaves of sets.

A morphism between stacks means a morphism in the ∞-category St_k. We then have the notion of monomorphisms (Definition B.9.5). We also have the notion of effective epimorphism (Definition 1.8.10) since St_k is an ∞-topos.

Definition 2.1.3. A *monomorphism* of stacks is defined to be a monomorphism in St_k. An *epimorphism* of stacks is defined to be an effective epimorphism in St_k.

Remark 2.1.4. We can describe monomorphisms and epimorphisms in St_k more explicitly. For a stack $\mathcal{X} \in \mathsf{St}_k$, we denote by $\pi_0(\mathcal{X}) \in \mathrm{Fun}((\mathsf{Aff}_k)^{\mathrm{op}}, \mathrm{Set})$ the sheaf of sets obtained by taking π_0. Here $(\mathsf{Aff}_k)^{\mathrm{op}}$ is regarded as an ordinary category. For a morphism $f : \mathcal{X} \to \mathcal{Y}$ of stacks, we denote by $\pi_0(f)$ the induced morphism $\pi_0(\mathcal{X}) \to \pi_0(\mathcal{Y})$ of sheaves of sets. Then

- By Fact B.9.7, f is a monomorphism if and only if the induced morphism $\pi_0(\Delta_f)$ of $\Delta_f : \mathcal{X} \to \mathcal{X} \times_{f,\mathcal{Y},f} \mathcal{X}$ is an isomorphism in $\mathrm{Fun}((\mathsf{Aff}_k)^{\mathrm{op}}, \mathrm{Set})$.
- f is an effective epimorphism if and only if the induced morphism $\pi_0(f) : \pi_0(\mathcal{X}) \to \pi_0(\mathcal{Y})$ is an epimorphism in the category $\mathrm{Fun}((\mathsf{Aff}_k)^{\mathrm{op}}, \mathrm{Set})$.

Note that the fiber product $\mathcal{X} \times_{f,\mathcal{Y},f} \mathcal{X}$ is the one in the ∞-categorical sense (Definition B.4.5). This description of monomorphisms and epimorphisms is consistent with [TVe2, Definition 1.3.1.2].

As in the ordinary Yoneda embedding $X \mapsto \mathrm{Hom}(-, X)$, we have the ∞-theoretic Yoneda embedding (Definition C.2.1)

$$j : \mathsf{Aff}_k \hookrightarrow \mathsf{St}_k.$$

Following the terminology in [TVe2], we introduce

Definition. A stack in the essential image of the Yoneda embedding j is called a *representable stack*.

We will often consider $\mathsf{Aff}_k \subset \mathsf{St}_k$ by the Yoneda embedding j and identify an affine scheme with the corresponding representable stack. We can transfer the ordinary notions on affine schemes to those on representable stacks. For example, we have

Definition 2.1.5. A *smooth morphism of representable stacks* is defined to be a smooth morphism of affine schemes in the sense of [EGA4, 4ème partie, Définition (17.3.1)].

Next we recall the notion of geometric stacks.

Definition 2.1.6 ([TVe2, Definition 1.3.3.1])**.** For $n \in \mathbb{Z}_{\geq -1}$, we define an *$n$-geometric stack*, an object in St_k, inductively on n. At the same time we also define an n-atlas of a stack, a *n-representable morphism* and a *n-smooth morphism* of stacks.

- Let $n = -1$.
  (i) A (-1)-*geometric stack* is defined to be a representable stack.
  (ii) A morphism $f : \mathcal{X} \to \mathcal{Y}$ of stacks is called (-1)-*representable* if for any representable stack U and any morphism $U \to \mathcal{Y}$ of stacks, the pullback $\mathcal{X} \times_{\mathcal{Y}} U$ is a representable stack.
  (iii) A morphism $f : \mathcal{X} \to \mathcal{Y}$ of stacks is called (-1)-*smooth* if it is (-1)-representable, and if for any representable stack U and any morphism $U \to \mathcal{Y}$ of stacks, the induced morphism $\mathcal{X} \times_{\mathcal{Y}} U \to U$ is a smooth morphism of representable stacks (Definition 2.1.5).
- Let $n \in \mathbb{N}$.
  (i) Let $\mathcal{X}$ be a stack. An *n-atlas of* $\mathcal{X}$ is a small family $\{U_i \to \mathcal{X}\}_{i \in I}$ of morphisms of stacks satisfying the following three conditions.
    – Each U_i is a representable stack.
    – Each morphism $U_i \to \mathcal{X}$ is $(n-1)$-smooth.
    – The morphism $\coprod_{i \in I} U_i \to \mathcal{X}$ is an epimorphism of stacks.
  We will sometimes denote an n-atlas $\{U_i \to \mathcal{X}\}_{i \in I}$ simply by $\{U_i\}_{i \in I}$.
  (ii) A stack $\mathcal{X}$ is called *n-geometric* if the following two conditions are satisfied.
    (a) The diagonal morphism $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is $(n-1)$-representable.
    (b) There exists an n-atlas of $\mathcal{X}$.
  (iii) A morphism $f : \mathcal{X} \to \mathcal{Y}$ of stacks is called *n-representable* if for any representable stack U and for any morphism $U \to \mathcal{Y}$ of stacks, the derived stack $\mathcal{X} \times_{\mathcal{Y}} U$ is n-geometric.
  (iv) A morphism $f : \mathcal{X} \to \mathcal{Y}$ of stacks is called *n-smooth* if for any representable stack U and any morphism $U \to \mathcal{Y}$ of stacks, there exists an n-atlas $\{U_i\}_{i \in I}$ of $\mathcal{X} \times_{\mathcal{Y}} U$ such that for each $i \in I$ the composition $U_i \to \mathcal{X} \times_{\mathcal{Y}} U \to U$ is a smooth morphism of representable stacks (Definition 2.1.5).

2.1.2. *Relation to algebraic stacks.* As an illustration of geometric stacks, let us explain the relation to algebraic spaces and algebraic stacks in the ordinary sense following [TVe2, §2.1.2]. We begin with

Definition 2.1.7 ([TVe2, §2.1.1]). (1) Let $m \in \mathbb{N}$. A stack $\mathcal{X} \in \mathsf{St}_k$ is called *m-truncated* if $\pi_j(\mathcal{X}(U), p)$ is trivial for any $U \in \mathsf{Aff}_k$, any $p \in \pi_0(\mathcal{X}(U))$ and any $j > m$.
(2) An *Artin m-stack* is an m-truncated n-geometric stack for some $n \in \mathbb{N}$.

Here we used the homotopy groups $\pi_i(S)$ of a simplicial set S (see [GJ, §1.7] for example). Let us cite a relation between truncation and geometricity.

Fact ([TVe2, Lemma 2.1.1.2]). An n-geometric stack is $(n+1)$-truncated.

These higher Artin stacks will be the ∞-theoretic counterpart of algebraic spaces and algebraic stacks. See the appendix §A.2 for the relevant definitions. Let us now cite from [TVe2, §2.1.2] a general construction of an embedding of the theory of fibered categories in groupoids into the theory of simplicial presheaves.

Let S be a site. The category Grpd/S of fibered categories in groupoids over S has a model structure such that the fibrant objects are ordinary stacks over S and the weak equivalences are the functors of fibered categories inducing equivalences on the associated stacks. The homotopy category Ho(Grpd/S) of this model category can be described as follows.

- The objects are ordinary stacks over S.
- The morphisms are 1-morphisms of ordinary stacks up to 2-isomorphisms.

By [TVe2, §2.1.2] we have a Quillen adjunction

$$\mathrm{Grpd/S} \rightleftarrows \mathrm{Sh}^{\leq 1}(\mathrm{S}),$$

where $\mathrm{Sh}^{\leq 1}(\mathrm{S})$ denotes the category of sheaves of simplicial sets over S with the 1-truncated local projective model structure [TVe1, Theorem 3.7.3].

Let us take S in the above argument to be the site $(\mathrm{Aff}_k, \mathrm{et})$ of affine schemes with the étale topology over a commutative ring k (Definition A.1.1). Then the above Quillen adjunction yields an adjunction of the underlying ∞-categories (Definition B.6.2) by Fact B.6.3.

Notation 2.1.8. Let us denote the obtained adjunction by

$$a : \mathsf{N}_{\mathrm{spl}}(\mathrm{Grpd}/(\mathrm{Aff}_k, \mathrm{et})) \rightleftarrows \mathsf{St}_k : t.$$

Algebraic stacks over k in the sense of Definition A.2.3 belong to $\mathrm{Grpd}/(\mathrm{Aff}_k, \mathrm{et})$. In particular, algebraic spaces and schemes over k belong there.

Now we have the following result.

Fact 2.1.9 ([TVe2, Proposition 2.1.2.1])**.** (1) If X is a scheme or an algebraic space over k, then $a(X)$ is an Artin 0-stack which is 1-geometric.
(2) If X is an algebraic stack over k, then $a(X)$ is an Artin 1-stack which is 1-geometric.

Hereafter we consider algebraic stacks, algebraic spaces and schemes as objects of St_k, i.e., as stacks.

Remark 2.1.10. The paper [O1] treats a slightly generalized notion of algebraic stacks. The difference is that in [O1] the diagonal morphism Δ is only assumed to be quasi-separated [EGA4, 1re Partie, Définition (1.2.1)]. The statement in Fact 2.1.9 (2) still holds for an algebraic stack in this sense.

The following Fact 2.1.11 shows that the notion of Artin n-stacks is a natural extension of the ordinary notion of schemes, algebraic spaces and algebraic stacks (see also Fact A.2.5).

Fact 2.1.11 ([TVe2, Remark 2.1.1.5])**.** Let $\mathcal{X}$ be an Artin m-stack with $m \in \mathbb{N}$.
(1) $\mathcal{X}$ is an algebraic space if and only if the following two conditions hold.
- $\mathcal{X}$ is a Deligne-Mumford m-stack, i.e., there exists an m-atlas $\{U_i\}_{i \in I}$ of $\mathcal{X}$ such that each morphism $U_i \to \mathcal{X}$ of stacks is an étale morphism.
- The diagonal map $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is a monomorphism of stacks.

If $\mathcal{X}$ is an algebraic space, then $\mathcal{X}$ is 1-geometric.
(2) $\mathcal{X}$ is a scheme if and only if there exists an m-atlas $\{U_i\}_{i \in I}$ of $\mathcal{X}$ such that each morphism $U_i \to \mathcal{X}$ of stacks is a monomorphism. If so then $\mathcal{X}$ is 1-geometric.
(3) Assume $\mathcal{X}$ is an algebraic space or a scheme. Then $\mathcal{X}$ has an affine diagonal, i.e., the diagonal map is (-1)-representable, if and only if it is 0-geometric.

Remark 2.1.12. By Fact 2.1.9, if $\mathcal{X}$ in Fact 2.1.11 is an algebraic space or a scheme, then we have $m = 0$ and $\mathcal{X}$ is 1-geometric. Thus we can depict the relation between sub-∞-categories of St and subcategories of the category $\mathrm{St}_{\mathrm{ord}}$ of ordinary stacks as

$$\begin{array}{ccccccccc}
\mathsf{N}(\mathrm{Sch}) & \hookrightarrow & \mathsf{N}(\mathrm{AS}) & \hookrightarrow & \mathsf{N}(\mathrm{AlgSt}) & & \hookrightarrow & & \mathsf{N}(\mathrm{St}_{\mathrm{ord}}) \\
\| & & \| & & \downarrow & & & & \downarrow a \\
\mathsf{Sch} & \hookrightarrow & \mathsf{AS}_{n=1}^{m=0} & \hookrightarrow & \mathsf{Artin}_{n=1}^{m=1} & \hookrightarrow & \mathsf{Artin}_{n=1} & \hookrightarrow \ \mathsf{Artin} = \mathrm{h}\,\mathsf{St}_{\mathrm{geom}}^{\mathrm{trun}} \ \hookrightarrow & \mathsf{St}
\end{array}$$

Here AS and AlgSt denote the category of algebraic spaced and of algebraic stacks respectively. The subscript n in the second row denotes the geometricity of a stack, and the superscript m denotes the truncation degree.

We close this part with the recollection of a quotient stack, a standard example of algebraic stacks.

Fact ([O2, Example 8.1.12])**.** Let S be a scheme, X be an algebraic space over S, and G be a smooth group scheme over S. Assume G acts on X. Define $[X/G]$ to be the ordinary stack on S (Definition A.2.1) whose objects are triples (T, P, π) consisting of the following data.
- $T \in \mathrm{Sch}_S$.
- P is a $G \times_S T$-torsor on the big étale site $\mathrm{ET}(T)$ (Definition A.1.1).
- $\pi : P \to X \times_S T$ is a $G \times_S T$-equivariant morphism of sheaves on Sch_T.

Then $[X/G]$ is an algebraic stack. A smooth covering of $[X/G]$ is given by $q : X \to [X/G]$, which is defined by the triple (S, G_X, ρ) with $G_X := G \times X$ the trivial G-torsor on X and $\rho : G_X \to X$ the action map of G on X.

The algebraic stack $[X/G]$ is called the *quotient stack*.

Definition 2.1.13. For a smooth group scheme G over a scheme S, the *classifying stack* BG of G is defined to be the quotient stack $[S/G]$ where G acts trivially on S.

In the case $S = \operatorname{Spec} k$, the classifying stack BG is a 1-geometric Artin 1-stack by Fact 2.1.9.

2.2. **Derived stacks.** In this subsection we present a summary of derived stacks. As in the previous §2.1, our exposition will be given in the ∞-categorical language.

2.2.1. *The ∞-category of simplicial modules.* This part is a preliminary for arguments in a simplicial setting.

Let k be a commutative ring. We denote by Mod_k the category of k-modules. We also denote by sMod_k the category of simplicial k-modules, i.e., of functors $\mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Mod}_k$. The category sMod_k is equipped with the model structure given by the Kan model structure on the underlying simplicial sets. We denote by $\mathrm{sMod}_k^{\circ} \subset \mathrm{sMod}_k$ the full subcategory of fibrant-cofibrant objects. Now recall the underlying ∞-category of a simplicial model category (Definition B.6.2). Using this notion, we introduce

Definition. The underlying ∞-category

$$\mathsf{sMod}_k := \mathsf{N}_{\mathrm{spl}}(\mathrm{sMod}_k^{\circ})$$

is called the *∞-category of simplicial k-modules.*

Let us give another description of the ∞-category sMod_k. We will use basic notions on dg-categories (see §9.1 for a short account). We denote by $\mathrm{C}^-(k)$ the dg-category of non-positively graded complexes of k-modules. Taking the dg-nerve (Definition D.3.1), we have an ∞-category $\mathsf{N}_{\mathrm{dg}}(\mathrm{C}^-(k))$. We then have an equivalence of ∞-categories

$$\mathsf{N}_{\mathrm{dg}}(\mathrm{C}^-(k)) \simeq \mathsf{sMod}_k.$$

This equivalence is given by the Dold-Kan correspondence [GJ, Chap. III, §2], [Lur2, Theorem 1.2.3.7].

The standard tensor product on $\mathrm{C}^-(k)$ induces an ∞-operad structure on sMod_k in the sense of [Lur2]. We will often denote the obtained ∞-operad $\mathsf{sMod}_k^{\otimes}$ by the simplified symbol sMod_k.

2.2.2. *Derived rings.* Following [T6, §2.2] we introduce the ∞-category of affine derived schemes.

Let us fix a commutative ring k. As in the previous §2.1, we denote by Com_k the category of commutative k-algebras.

Definition 2.2.1. A simplicial object in Com_k, in other words, a functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Com}_k$, is called a *simplicial commutative k-algebra* or a *derived k-algebra*. In the case $k = \mathbb{Z}$, we call it a *derived ring*. We denote by sCom_k the category of derived k-algebras.

We can associate to a derived k-algebra A a commutative graded k-algebra

$$\pi_*(A) = \oplus_{n\in\mathbb{N}}\pi_n(A),$$

where the base point is taken to be $0 \in A$. See [GJ, §1.7] for the definition of homotopy groups of simplicial sets. In particular, we have a commutative k-algebra $\pi_0(A)$.

The category sCom_k has a model structure induced by the Kan model structure (Fact B.1.2) of the underlying simplicial sets. Then, regarding sCom_k as a simplicial model category (Definition B.6.1), we have the underlying ∞-category $\mathsf{N}_{\mathrm{spl}}(\mathrm{sCom}_k^{\circ})$ (Definition B.6.2). Here $\mathrm{sCom}_k^{\circ} \subset \mathrm{sCom}_k$ is the full subcategory spanned by fibrant-cofibrant objects with respect to the Kan model structure, and $\mathsf{N}_{\mathrm{spl}}(-)$ is the simplicial nerve construction (Definition 1.4.1).

Definition 2.2.2. The obtained ∞-category

$$\mathsf{sCom}_k := \mathsf{N}_{\mathrm{spl}}(\mathrm{sCom}_k^{\circ})$$

is called the *∞-category of derived k-algebras.*

Remark. Here is another description of sCom_k. Consider the set W of weak equivalences in the model category sCom_k in the above sense. We apply the ∞-localization (Definition B.7.1) to the pair $(\mathsf{N}(\mathrm{sCom}_k), W)$, and have an equivalence of ∞-categories

$$\mathsf{sCom}_k \simeq \mathsf{N}(\mathrm{sCom}_k)[W^{-1}].$$

Remark 2.2.3. We have a functor

$$\mathsf{sCom}_k \longrightarrow \mathsf{CAlg}_k^{\mathrm{cn}}$$

of ∞-categories, which is an equivalence if k is a $\mathbb{Q}$-algebra. The target $\mathsf{CAlg}_k^{\mathrm{cn}}$ is the ∞-category of connected *commutative ring spectra*, which is a foundation of Lurie's spectral algebraic geometry [Lur5]–[Lur14]. See E.2 for the detail.

Definition 2.2.4. (1) The ∞-category dAff_k of *affine derived schemes over k* is defined to be

$$\mathsf{dAff}_k := (\mathsf{sCom}_k)^{\mathrm{op}},$$

the opposite ∞-category of sCom_k. The object in dAff_k corresponding to $A \in \mathsf{sCom}_k$ will be denoted by $\mathrm{dSpec}\, A$ and called the *affine derived scheme* of A. A morphism in dAff_k will be called a *morphism of affine derived schemes.*

(2) For an affine derived scheme $U = \mathrm{dSpec}\, A$ over k, the morphism $U \to \mathrm{dSpec}\, k$ of affine derived schemes corresponding to $k \to A$ is called the *structure morphism* of U.

In terms of the ∞-operad $\mathsf{sMod}_k^{\otimes}$ (§2.2.1), a derived k-algebra is nothing but a commutative ring object in $\mathsf{sMod}_k^{\otimes}$.

One may infer a relation between the ∞-categories dAff_k and Aff_k in the previous §2.1. We postpone the explanation to §2.2.5.

Let us now introduce some classes of morphisms of affine derived schemes.

Definition 2.2.5 ([TVe2, §2.2.2])**.** (1) A morphism $A \to B$ in sCom_k is *étale* (resp. *smooth*, resp. *flat*) if the following two conditions are satisfied.
- the induced morphism $\pi_0(A) \to \pi_0(B)$ is an étale (resp. smooth, resp. flat) morphism of commutative k-algebras,
- the induced morphism $\pi_i(A) \otimes_{\pi_0(A)} \pi_0(B) \to \pi_i(B)$ is an isomorphism for any i.

(2) A morphism $X \to Y$ in the ∞-category dAff_k of affine derived schemes is called *étale* (resp. *smooth*, resp. *flat*) if the corresponding one in sCom_k is so.

For later use we also introduce the notion of a finitely presented morphism. We need some terminology.
- By a *filtered system* $\{C_i\}_{i \in I}$ of objects in an ∞-category C, we mean a diagram in C indexed by a κ-filtered poset I with some regular cardinal κ.
- $\mathsf{C}_{C/}$ denotes the under-∞-category of an ∞-category C under an object $C \in \mathsf{C}$ (Definition B.3.1).
- $\mathrm{Map}_{\mathsf{C}}(-,-)$ denotes the mapping space for an ∞-category C (Definition B.2.4), which is an object of the homotopy category $\mathcal{H}$ of spaces (Definition 1.3.2).

Definition 2.2.6 ([TVe2, Definition 1.2.3.1], [TVa, §2.3])**.** (1) A morphism $f : A \to B$ in sCom_k is called *finitely presented* if for any filtered system $\{C_i\}_{i \in I}$ of objects in $(\mathsf{sCom}_k)_{A/}$ the natural morphism

$$\varinjlim_{i \in I} \mathrm{Map}_{(\mathsf{sCom}_k)_{A/}}(B, C_i) \longrightarrow \mathrm{Map}_{(\mathsf{sCom}_k)_{A/}}(B, \varinjlim_{i \in I} C_i)$$

is an isomorphism in $\mathcal{H}$.

(2) A derived k-algebra $A \in \mathsf{sCom}_k$ is called *finitely presented* or *of finite presentation* if the morphism $k \to A$ is finitely presented in the sense of (1).

2.2.3. *Derived stacks.* In [TVe2, Chap. 2.2] derived stacks are defined in terms of homotopical algebraic geometry context. Here we give a redefinition in terms of the theory of ∞-topos following [T6, §3.2].

We begin with giving dAff_k an ∞-categorical Grothendieck topology (§1.7).

Definition 2.2.7 ([TVe2, §2.2.2])**.** A family $\{\mathrm{dSpec}\, A_i \to \mathrm{dSpec}\, A\}_{i \in I}$ of morphisms in dAff_k is called an *étale covering* if the following conditions are satisfied.
- For each $i \in I$, the morphism $\pi_*(A) \otimes_{\pi_0(A)} \pi_0(A_i) \to \pi_*(A_i)$ is an isomorphism of graded rings.
- There exists a finite subset $J \subset I$ such that the induced morphism $\coprod_{j \in J} \mathrm{Spec}\, \pi_0(A_j) \to \mathrm{Spec}\, \pi_0(A)$ of affine schemes is a surjection.

Let us note that this definition is equivalent to the one in [TVe2, Definition 2.2.2.12] by the argument given there.

By the argument in [TVe2, §2.2.2], étale coverings give a Grothendieck topology on dAff_k in the sense of Definition 1.7.2. In particular, étale coverings are stable under pullbacks.

Definition. The obtained ∞-site is called the *étale ∞-site* and denoted by $(\mathsf{dAff}_k, \mathrm{et})$

Then, as in Definition 2.1.1, we can introduce

Definition 2.2.8. The *∞-category of derived stacks over k* is defined to be

$$\mathsf{dSt}_k := \mathsf{Sh}(\mathsf{dAff}_k, \mathrm{et}).$$

Its object is called a *derived stack.*

Remark. In [TVe2] the model category corresponding to dSt_k is denoted by $D^-\mathrm{St}(k)$ and its object is called a D^--stack. See also Remark 2.1.2 on the difference of our terminology on derived stacks and that on D^--stacks given in [TVe2].

The ∞-category dSt_k has similar properties to St_k. For example, by Fact 1.7.4 and Corollary C.3.5, we have

Fact 2.2.9. dSt_k is a hypercomplete ∞-topos (Definition C.3.1).

We also have

Fact 2.2.10 ([TVe2, Lemma 2.2.2.13])**.** The ∞-topos dSt_k is quasi-compact (Definition 1.8.12).

As in the case of stacks, we have the ∞-theoretic Yoneda embedding (Definition C.2.1)

$$j : \mathsf{dAff}_k \longhookrightarrow \mathsf{dSt}_k.$$

Definition. A derived stack in the essential image of j is called a *representable derived stack.*

Hereafter, regarding dAff_k as a sub-∞-category of dSt_k by the Yoneda embedding j, we consider an affine derived scheme as a derived stack. Thus every notion on affine derived schemes such as in Definition 2.2.5 can be transfered to that on representable derived stacks.

For each $\mathcal{X} \in \mathsf{dSt}_k$, we have a morphism $\mathcal{X} \to \mathrm{dSpec}\, k$ in dSt_k, which will be called the *structure morphism* of $\mathcal{X}$.

Let us introduce some notions on morphisms of derived stacks. A *morphism* between derived stacks means a morphism in the ∞-category St_k. We then have the notion of *monomorphisms* (Definition B.9.5). We also have the notion of *effective epimorphism* (Definition 1.8.10) since St_k is an ∞-topos.

Definition 2.2.11. A *monomorphism* or an *injection* of derived stacks is defined to be a monomorphism in dSt_k. A *epimorphism* or a *surjection* of derived stacks is defined to be an effective epimorphism in dSt_k.

Remark. As in the case of stacks (Remark 2.1.4), we can restate this definition more explicitly. For a derived stack $\mathcal{X}$, we denote by $\pi_0(\mathcal{X}) \in \mathrm{Fun}((\mathsf{dAff}_k)^{\mathrm{op}}, \mathrm{Set})$ the sheaf of sets obtained by taking π_0. For a morphism $f : \mathcal{X} \to \mathcal{Y}$ of derived stacks, we denote by $\pi_0(f)$ the induced morphism $\pi_0(\mathcal{X}) \to \pi_0(\mathcal{Y})$ of sheaves of sets. Then

- f is a monomorphism if and only if the induced morphism $\pi_0(\Delta_f)$ of $\Delta_f : \mathcal{X} \to \mathcal{X} \times_{f,\mathcal{Y},f} \mathcal{X}$ is an isomorphism in $\mathrm{Fun}((\mathsf{dAff}_k)^{\mathrm{op}}, \mathrm{Set})$.
- f is an epimorphism if and only if the induced morphism $\pi_0(f)$ is an epimorphism in the category $\mathrm{Fun}((\mathsf{dAff}_k)^{\mathrm{op}}, \mathrm{Set})$.

Again, our convention on monomorphisms and epimorphisms is consistent with [TVe2, Definition 1.3.1.2].

Let us also introduce

Definition 2.2.12 ([TVe2, Definition 1.3.6.4 (2)])**.** A morphism $f : \mathcal{X} \to \mathcal{Y}$ in dSt_k is *quasi-compact* if for any morphism $p : U \to \mathcal{Y}$ from an affine derived stack U, there exists a finite family $\{U_i\}_{i \in I}$ of affine derived stacks and a surjection $\coprod_{i \in I} U_i \to \mathcal{X} \times_{f,\mathcal{Y},p} U$.

In the most of the following sections we will work on a fixed commutative ring, but in §8 we need a base change. For that, let us consider a field k and an extension L of k. Then, for a derived stack $\mathcal{X} \in \mathsf{dSt}_k$, the fiber product $\mathcal{X} \times_{\mathrm{dSpec}\, k} \mathrm{dSpec}_L$ in dSt_k defines a derived stack over L.

Notation 2.2.13. We denote the fiber product $\mathcal{X} \times_{\mathrm{dSpec}\, k} \mathrm{dSpec}_L \in \mathsf{dSt}_L$ by $\mathcal{X}_L$ or $\mathcal{X} \otimes_k L$.

2.2.4. *Geometric derived stacks.* Following [TVe2, Chap. 2.2] and [TVa, §2.3] we recall the notion of geometric derived stack.

Definition 2.2.14 ([TVe2, Definition 1.3.3.1])**.** For $n \in \mathbb{Z}_{\geq -1}$, an *$n$-geometric stack* is an object in dSt_k defined in the same way as in Definition 2.1.6 with the replacement of "representable stack" by "representable derived stack". An *n-atlas*, an *n-representable morphism* and an *n-smooth morphism* of derived stacks are inductively defined in the same way.

Remark 2.2.15.
1. In the recursive definition of n-smoothness of morphisms, we use Definition 2.2.5 for smoothness of morphisms in dAff.
2. In [TVe2, §1.3.4] it is explained that the condition (b) in the definition of n-geometric derived stacks follows from the condition (a). Thus it is enough to assume the existence of n-atlas only.
3. In [T6] a geometric derived stack is called a derived Artin stack. We will not use this terminology.

By definition we can check the following statement (see also Fact 2.1.11 (3)).

Lemma 2.2.16. A derived stack $\mathcal{X}$ is 0-geometric if and only if it has an affine diagonal, i.e. the diagonal morphism $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is (-1)-representable.

We will repeatedly use the following properties.

Fact 2.2.17 ([TVe2, Proposition 1.3.3.3, Corollary 1.3.3.5])**.** Let $n \in \mathbb{Z}_{\geq -1}$.
1. An $(n-1)$-representable (resp. $(n-1)$-smooth) morphism is n-representable (resp. n-smooth). In particular, an $(n-1)$-geometric stack is n-geometric.
2. The class of n-representable (resp. n-smooth) morphisms are stable under isomorphisms, pullbacks and compositions.
3. For $n \in \mathbb{N}$, the ∞-category of n-geometric derived stacks is stable under pullbacks and taking small coproducts.

We next recall the following extended class of geometric derived stacks.

Definition 2.2.18 ([TVa, Definition 2.17]). A derived stack $\mathcal{X}$ is called *locally geometric* if $\mathcal{X}$ is equivalent to a filtered colimit

$$\mathcal{X} \simeq \varinjlim_{i \in I} \mathcal{X}_i$$

of derived stacks $\{\mathcal{X}_i\}_{i \in I}$ satisfying the following conditions.

- Each derived stack $\mathcal{X}_i$ is n_i-geometric for some $n_i \in \mathbb{Z}_{\geq -1}$.
- Each morphism $\mathcal{X}_i \to \mathcal{X}_i \times_{\mathcal{X}} \mathcal{X}_i$ of derived stacks induced by $\mathcal{X}_i \to \mathcal{X}$ is an equivalence in dSt_k.

Definition 2.2.19 ([TVa, §2.3]).
(1) An n-geometric derived stack $\mathcal{X}$ is called *locally of finite presentation* if it has an n-atlas $\{U_i\}_{i \in I}$ such that for each representable derived stack $U_i \simeq \operatorname{dSpec} A_i$ the derived k-algebra A_i is finitely presented (Definition 2.2.6).
(2) A locally geometric derived stack $\mathcal{X}$ is *locally of finite presentation* if writing $\mathcal{X} \simeq \varinjlim_i \mathcal{X}_i$ each geometric derived stack $\mathcal{X}_i$ can be chosen to be locally of finite presentation in the sense of (1).

The class of locally geometric derived stacks locally of finite presentation contains the moduli stack of perfect dg-modules by the following theorem of Toën and Vaquié [TVa]. This fact is very important for us and will be explained in detail in §9.4.

Fact 2.2.20 ([TVa, Theorem 3.6]). For a dg-category D over k of finite type, we have the moduli space $\mathcal{M}(\mathrm{D})$ of perfect D^{op}-modules, which is a locally geometric derived stack locally of finite presentation.

For later use, we record

Definition 2.2.21 ([TVe2, Definition 1.3.6.2, Lemma 2.2.3.4]). Let $\mathbf{Q}$ be either of the following properties of morphisms in dAff_k:

$$\mathbf{Q} = \text{flat, smooth, étale, locally of finitely presented.}$$

A morphism $f : \mathcal{X} \to \mathcal{Y}$ of derived stacks *has property* $\mathbf{Q}$ if it is n-representable for some n and if for any affine derived scheme U and any morphism $U \to \mathcal{Y}$ there exists an n-atlas $\{U_i\}_{i \in I}$ of $\mathcal{X} \times_{\mathcal{Y}} U$ such that each morphism $U_i \to \mathcal{X} \times_{\mathcal{Y}} U \to U$ in dAff_k has property $\mathbf{Q}$.

In this definition the choice of n is irrelevant by [TVe2, Proposition 1.3.3.6]. We also note that for $\mathbf{Q}$ = smooth, Definition 2.2.21 of an n-representable morphism being a smooth morphism and Definition 2.2.14 of an n-smooth morphism are compatible.

Fact 2.2.22 ([TVe2, Proposition 1.3.6.3]). Let $\mathbf{Q}$ be one of the properties of morphisms in Definition 2.2.21.
(1) Morphisms of derived stacks having property $\mathbf{Q}$ are stable under equivalences, compositions and pullbacks.
(2) Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of n-geometric derived stacks. If there is an n-atlas $\{U_i \to \mathcal{Y}\}_{i \in I}$ such that each projection $\mathcal{X} \times_{\mathcal{Y}} U_i \to U_i$ has property $\mathbf{Q}$, then f has property $\mathbf{Q}$.

Next we introduce the relative dimension of a smooth morphism. One can check the following is well-defined using Fact 2.2.22 and Definition 2.2.5.

Definition 2.2.23. A morphism $f : \mathcal{X} \to \mathcal{Y}$ of derived stacks is *smooth of relative dimension* d if it satisfies the following condition.

- For any $U \in \mathsf{dAff}_k$ and any morphism $U \to \mathcal{Y}$ of derived stacks, take an n-atlas $\{U_i\}_{i \in I}$ of $\mathcal{X} \times_{\mathcal{Y}} U$ as Definition 2.2.21. Denote by $g : U_i \to \mathcal{X} \times_{\mathcal{Y}} U \to U$ the smooth morphism in dAff_k. Then $\pi_0(g) : \pi_0(U_i) \to \pi_0(U)$ is smooth of relative dimension d in the scheme-theoretic meaning [EGA4, Chap. IV, §17.10].

Let us also introduce immersions of derived stacks.

Definition 2.2.24 ([TVe2, Definition 2.2.3.5]).
(1) A morphism of derived stacks is an *open immersion* if it is locally of finite presentation (Definition 2.2.21), flat (Definition 2.2.21) and a monomorphism (Definition 2.2.11).
(2) A morphism $\mathcal{F} \to \mathcal{X}$ of derived stacks is a *closed immersion* if it is (-1)-representable and if for any representable stack $U \simeq \operatorname{dSpec} A$ and any morphism $U \to \mathcal{X}$ the morphism $\mathcal{F} \times_{\mathcal{X}} U \simeq \operatorname{dSpec} B \to U \simeq \operatorname{dSpec} A$ induces an epimorphism $\pi_0(A) \to \pi_0(B)$ of rings.

Note that an open immersion in the above sense is called a Zariski open immersion in [TVe2]. One can check by definition that a closed immersion $i : \mathcal{F} \to \mathcal{X}$ of derived stacks determines an open immersion $\mathcal{U} \to \mathcal{X}$ with the property $[\mathcal{U}(T)] = [\mathcal{X}(T)] \setminus [\mathcal{F}(T)]$ for any affine derived scheme T, where $[-]$ denotes the homotopy type of a simplicial set (Definition B.2.1). Moreover $\mathcal{U}$ is unique up to contractible ambiguity.

Notation 2.2.25. We call the open immersion $\mathcal{U} \to \mathcal{X}$ the *complement* of $i : \mathcal{F} \to \mathcal{X}$.

We finally have

Definition 2.2.26. A morphism of derived stacks is *of finite presentation* if it is locally of finite presentation (Definition 2.2.21) and quasi-compact (Definition 2.2.12).

2.2.5. *Truncation.* Let us now explain the relationship of higher Artin stacks in §2.1.2 and geometric derived stacks. We follow the argument in [TVe2, §§2.1.1, 2.1.2, 2.2.4].

Recall the ∞-category Com_k of commutative k-algebras (see §2.1). Regarding commutative k-algebras as constant derived k-algebras, we have an embedding $\mathsf{Com}_k \to \mathsf{sCom}_k$ of ∞-category. Taking the opposites, we have the corresponding embedding $i : \mathsf{Aff}_k \hookrightarrow \mathsf{dAff}_k$. It gives rise to an adjunction

$$\pi_0 : \mathsf{dAff}_k \rightleftarrows \mathsf{Aff}_k : i$$

of ∞-categories (Definition B.5.2), where π_0 denotes the functor taking π_0 of derived k-algebras. This adjunction naturally extends to a new adjunction $\mathsf{PSh}(\mathsf{Aff}_k) \rightleftarrows \mathsf{PSh}(\mathsf{dAff}_k)$ of the ∞-categories of presheaves (Definition 1.5.1). In [TVe1, §4.8] the corresponding Quillen adjunction is denoted by $i_! : \mathrm{PSh}(\mathrm{Aff}_k) \rightleftarrows \mathrm{PSh}(\mathrm{dAff}_k) : i^*$.

By the argument of [TVe2, §2.2.4], the last adjunction $(i_!, i^*)$ yields a Quillen adjunction between simplicial categories, whose underlying adjunction on the homotopy categories can be expressed as $\mathrm{h}\,\mathsf{Sh}(\mathsf{Aff}_k, \mathrm{et}) \rightleftarrows \mathrm{h}\,\mathsf{Sh}(\mathsf{dAff}_k, \mathrm{et})$. Then by Fact B.6.3, we can construct from the Quillen adjunction an adjunction of ∞-categories. Let us denote it by

$$\mathrm{Dex} : \mathsf{St}_k = \mathsf{Sh}(\mathsf{Aff}_k, \mathrm{et}) \rightleftarrows \mathsf{dSt}_k = \mathsf{Sh}(\mathsf{dAff}_k, \mathrm{et}) : \mathrm{Trc}\,.$$

Fact ([TVe2, Lemma 2.2.4.1])**.** The functor $\mathrm{Dex} : \mathsf{St}_k \to \mathsf{dSt}_k$ is fully faithful.

Definition 2.2.27 ([TVe2, Definition 2.2.4.3])**.**
1. Trc : $\mathsf{dSt}_k \to \mathsf{St}_k$ is called the *truncation* functor and $\mathrm{Dex} : \mathsf{St}_k \to \mathsf{dSt}_k$ is called the *extension* functor.
2. A derived stack $\mathcal{X} \in \mathsf{dSt}_k$ is called *truncated* if the adjunction $\mathrm{Dex}(\mathrm{Trc}(\mathcal{X})) \to \mathcal{X}$ is an isomorphism in $\mathrm{h}\,\mathsf{dSt}_k$.

As noted in [TVe2], Trc commutes with limits and colimits, and Dex is fully faithful and commutes with colimits, but Dex does not commute with limits. However we have the following results.

Fact 2.2.28 ([TVe2, Proposition 2.2.4.4])**.**
1. The truncation functor Trc sends an n-geometric derived stack to an n-geometric stack and a flat (resp. smooth, resp. étale) morphism between n-geometric derived stacks to a morphism of the same type between n-geometric stacks. The functor Trc also sends epimorphisms of derived stacks to epimorphisms of stacks.
2. Dex sends n-geometric stacks to n-geometric derived stacks, pullbacks of n-geometric stacks to those of n-geometric derived stacks, and flat (resp. smooth, resp. étale) morphisms of n-geometric stacks to those of n-geometric derived stacks.

Recall that an algebraic stack over k (Definition A.2.3) belongs to the category $\mathrm{h}(\mathrm{Grpd}/(\mathrm{Aff}_k, \mathrm{et}))$ and that we have the fully faithful functor a (Notation 2.1.8). Now we introduce

Definition 2.2.29. Consider the composition

$$\iota := \mathrm{Dex} \circ a : \mathsf{N}_{\mathrm{spl}}(\mathrm{Grpd}/(\mathrm{Aff}_k, \mathrm{et})) \longrightarrow \mathsf{St}_k \longrightarrow \mathsf{dSt}_k.$$

For an algebraic stack X, we call the image $\iota(X)$ the *derived stack associated to* X.

By Fact 2.1.9, Remark 2.1.10 and Fact 2.2.28, we have

Lemma 2.2.30. Let X be an algebraic stack. Then the derived stack $\iota(X)$ is truncated and 1-geometric (Definition 2.2.14).

Remark 2.2.31. As in Remark 2.1.12, we can depict the relation between sub-∞-categories of St and sub-∞-categories of dSt as

$$\begin{array}{ccccccccccc}
\mathsf{Sch} & \hookrightarrow & \mathsf{AS}^{m=0}_{n=1} & \hookrightarrow & \mathsf{Artin}^{m=1}_{n=1} & \hookrightarrow & \mathsf{Artin}_{n=1} & \hookrightarrow & \mathsf{Artin} = \mathsf{St}^{\mathrm{trun}}_{\mathrm{geom}} & \hookrightarrow & \mathsf{St} \\
 & & \big\downarrow & & \big\downarrow\wr & & \big\downarrow\wr & & \big\downarrow\wr & & \big\downarrow \mathrm{Dex} \\
 & & \mathsf{dSt}^{m=0}_{n=1} & \hookrightarrow & \mathsf{dSt}^{m=1}_{n=1} & \hookrightarrow & \mathsf{dSt}^{\mathrm{trun}}_{n=1} & \hookrightarrow & \mathsf{dSt}^{\mathrm{trun}}_{\mathrm{geom}} & \hookrightarrow & \mathsf{dSt}
\end{array}$$

One also can check that the notions ordinary algebraic stacks (§A.2) are compatible with those on derived stacks under the functor Dex. As for the properties of morphisms, we collect the corresponding definitions in the following table. We will give an additional table after introducing derived algebraic spaces (Remark 4.2.13)

property of morphisms	derived stacks	algebraic stacks
surjective	Definition 2.2.11	Definition A.2.7
locally of finite presentation	Definition 2.2.21	Definition A.2.7
étale	Definition 2.2.21	Definition A.2.8
smooth of relative dimension d	Definition 2.2.23	Definition A.2.8
quasi-compact	Definition 2.2.12	Definition A.2.8
closed immersion	Definition 2.2.24	Definition A.2.8

TABLE 2.1. Morphisms between derived and algebraic stacks

2.2.6. *Derived stacks of quasi-coherent modules and vector bundles.* We close this subsection by giving some examples of geometric derived stacks following [TVe2, §1.3.7, §2.2.6.1]. Let k be a commutative ring as before.

For a derived k-algebra A, an *A-module* means an A-module object in the ∞-operad $\mathsf{sMod}_k^{\otimes}$ (§2.2.1). A-modules form an ∞-category denoted by sMod_A. It has an induced ∞-operad structure from $\mathsf{sMod}_k^{\otimes}$, and the obtained ∞-operad is denoted by $\mathsf{sMod}_A^{\otimes}$. *Derived A-algebras* mean commutative ring objects in $\mathsf{sMod}_A^{\otimes}$, which form an ∞-category denoted by sCom_A.

For a derived k-algebra A, the category $\mathrm{QCoh}(A)$ of quasi-coherent modules on A is defined as

- An object is a data (M, α) consisting of a family $M = \{M_B\}_B$ of B-modules for $B \in \mathsf{sCom}_A$ and a family $\alpha = \{\alpha_u : M_B \otimes_B B' \to M'_B\}_u$ of isomorphisms for morphisms $u : B \to B'$ in sCom_A such that $\alpha_v \circ (\alpha_u \otimes_{B'} B'') = \alpha_{v \circ u}$ for any composable morphisms $B \xrightarrow{u} B' \xrightarrow{v} B''$.
- A morphism $f : (M, \alpha) \to (M', \alpha')$ is a family $f = \{f_B : M_B \to M_{B'}\}_B$ of morphisms of B-modules for $B \in \mathsf{sCom}_A$ such that for any $u : B \to B'$ in sCom_A we have $f_{B'} \circ \alpha_u = \alpha'_u \circ (f_B \otimes_B B') : M_B \otimes_B B' \to M'_{B'}$.

The category $\mathrm{QCoh}(A)$ inherits a model structure induced from Kan model structure of simplicial sets. Let us now consider the subcategory $\mathrm{QCoh}(A)^c_W \subset \mathrm{QCoh}(A)$ consisting of equivalences between cofibrant objects. Taking the nerve, we have an ∞-category $\mathsf{QCoh}(A) := \mathsf{N}(\mathrm{QCoh}(A)^c_W)$. It enjoys the property $\pi_1(\mathsf{QCoh}(A), M) \simeq \mathrm{Aut}_{\mathrm{h}\,\mathsf{sMod}_A}(M)$. Now we cite

Fact ([TVe2, Theorem 1.3.7.2])**.** The correspondence $A \mapsto \mathsf{QCoh}(A)$ determines a derived stack

$$\mathcal{QC}oh : \mathsf{sCom} = (\mathsf{dAff})^{\mathrm{op}} \longrightarrow \mathcal{S}.$$

Next we turn to the definition of vector bundles.

Definition 2.2.32. Let A be a derived k-algebra, and let $r \in \mathbb{N}$. An A-module $B \in \mathsf{sMod}_A$ is called a *rank r vector bundle* if there exists an étale covering $A \to A'$ (Definition 2.2.7) such that $M \otimes_A A' \simeq (A')^r$ in $\mathrm{h}\,\mathsf{sMod}_{A'}$.

We have the corresponding full sub-∞-category $\mathsf{Vect}_r \subset \mathsf{QCoh}(A)$ and the *derived stack $\mathcal{V}ect_r$ of rank r vector bundles.*

Recall the classifying stack BG for a smooth group scheme G over a scheme S (Definition 2.1.13). It is an algebraic stack, so that we have the corresponding truncated derived stack $\iota(BG)$.

In the case $G = \mathrm{GL}_r$, regarded as a group scheme over k, the classifying stack $B\,\mathrm{GL}_r$ is nothing but the moduli space of rank r vector bundles. Now we have

Fact 2.2.33 ([TVe2, §2.2.6.1])**.** We have

$$\mathcal{V}ect_r \simeq \iota(B\,\mathrm{GL}_r).$$

In particular, $\mathcal{V}ect_r$ is a truncated (Definition 2.2.27 (3)) 1-geometric derived stack with (-1)-representable diagonal. It is also locally of finite presentation (Definition 2.2.19).

Note that in [TVe2, §2.2.6.1] the word “affine diagonal” is used for “(-1)-representable diagonal”.

3. Sheaves on ringed ∞-topoi

This section is a preliminary for our study of sheaves on derived stacks. In a general setting of ringed ∞-topoi, we introduce the notions of sheaves of modules and functors between them. Some of them are already given in Lurie's work [Lur2, Lur5, Lur7]. These will give us an ∞-theoretical counterpart of the theory of (unbounded) derived categories ([Sp], [KS, Chap. 18]).

We manly focus on sheaves of *stable* modules, so that the resulting ∞-category will be stable in the sense of [Lur2, Chap. 1]. A stable ∞-category is an ∞-theoretical counterpart of a triangulated category. Thus our ∞-category of sheaves of stable modules will be an ∞-categorical counterpart of the derived category of sheaves of modules.

3.1. **Geometric morphisms of ∞-topoi.** In this subsection we give some complementary explanation on ∞-topoi.

Since an ∞-topos is an ∞-category, we automatically have

Definition. A morphism or a functor of ∞-topoi is defined to be a functor of ∞-categories (Definition 1.2.3).

However, this definition is poor as the correct notion of morphisms between (ordinary) topoi implies. Recall that a morphism $f : T \to T'$ of topoi is defined to be an adjunction $f^* : T' \rightleftarrows T : f_*$ [SGA4, IV.7]. In this subsection we recall the notion of *geometric morphisms*, which is the genuine notion of morphisms between ∞-topoi.

Definition 3.1.1. A morphism $f_* : \mathsf{T} \to \mathsf{T}'$ of ∞-topoi is *geometric* if it admits a left adjoint $f^* : \mathsf{T}' \to \mathsf{T}$ which is left exact (Definition B.10.1). In this case, the resulting adjunction will be denoted by

$$f^* : \mathsf{T}' \rightleftarrows \mathsf{T} : f_*.$$

Remark 3.1.2. Since either of f_* and f^* determines the other up to contractible ambiguity, we sometimes refer to f^* as a geometric morphism. Following [Lur1, Remark 6.3.1.7], we always denote a left adjoint by upper asterisk such as f^*, and denote a right adjoint by lower asterisk such as f_*.

By [Lur1, Remark 6.3.1.2] any equivalence of ∞-topoi, which is defined to be an equivalence as ∞-categories, is a geometric morphism. Also, by [Lur1, Remark 6.3.1.3], the class of geometric morphisms is stable under composition.

We now introduce the ∞-category RTop of ∞-categories and geometric morphisms.

Definition 3.1.3. The ∞-category RTop is given by the following description.
- The objects are (not necessarily small) ∞-topoi.
- The morphisms are functors $f_* : \mathsf{T} \to \mathsf{T}'$ of ∞-topoi which has a left exact left adjoint.

We also have the ∞-category LTop whose objects are the same as RTop but morphisms are functors $f^* : \mathsf{T}' \to \mathsf{T}$ preserving small colimits and finite limits. By [Lur1, Corollary 6.3.1.8], we have an equivalence $\mathsf{LTop} \simeq \mathsf{RTop}^{\mathrm{op}}$. Here is a list of some properties of RTop.

Fact 3.1.4.
(1) RTop admits small colimits [Lur1, Proposition 6.3.2.1].
(2) RTop admits small limits [Lur1, Corollary 6.3.4.7].
(3) The ∞-category $\mathcal{S}$ of spaces is a final object of RTop [Lur1, Proposition 6.3.3.1].

Recall that for an ∞-topos T and $U \in \mathsf{T}$, the over-∞-category $\mathsf{T}_{/U}$ is an ∞-topos (Fact 1.8.6). It is equipped with the following geometric morphisms.

Fact 3.1.5 ([Lur1, Proposition 6.3.5.1 (2)])**.** For $U \in \mathsf{T}$, the canonical functor $j_! : \mathsf{T}_{/U} \to \mathsf{T}$ of the over-∞-category $\mathsf{T}_{/U}$ (Corollary B.3.2) has a right adjoint j^* which commutes with colimits. Thus j^* has a right adjoint j_*, and we have two geometric morphisms of ∞-topoi:

$$j_! : \mathsf{T}_{/U} \rightleftarrows \mathsf{T} : j^*, \quad j^* : \mathsf{T} \rightleftarrows \mathsf{T}_{/U} : j_*.$$

Following the standard terminology in [SGA4], we call the triple $(j_!, j^*, j_*)$ *biadjunction.* We close this part by

Notation 3.1.6. Given an ∞-topos T, $U \in \mathsf{T}$ and $\mathcal{F} \in \mathsf{Shv}_{\mathsf{C}}(\mathsf{T})$ for some ∞-category C admitting small limits, we denote $\mathcal{F}|_U := j^*(\mathcal{F})$ and call it the *restriction of* $\mathcal{F}$ *to* U.

3.2. **Ringed ∞-topoi.**

3.2.1. *Definition.* We will use the language of ∞-operad [Lur2] in the following. Let C be a symmetric monoidal ∞-category [Lur2, §2.1]. Then we have the notion of *commutative ring objects* in C, and they form an ∞-category $\mathsf{CAlg}(\mathsf{C})$.

Remark 3.2.1. Let us give a more strict explanation. Let $\mathbb{E}_\infty^\otimes = \mathrm{Comm}^\otimes$ be the commutative ∞-operad [Lur2, §2.1.1, §5.1.1]. Then we have the ∞-category $\mathsf{Alg}_{\mathbb{E}_\infty}(\mathsf{C})$ of $\mathbb{E}_\infty$-algebra objects in C [Lur2, §3.1]. In the above we denoted $\mathsf{CAlg}(\mathsf{C}) := \mathsf{Alg}_{\mathbb{E}_\infty}(\mathsf{C})$. Let $R \in \mathsf{Alg}_{\mathbb{E}_\infty}(\mathsf{C})$.

Let T be an ∞-topos. Consider the ∞-category $\mathsf{Shv}_{\mathsf{CAlg}(\mathsf{C})}(\mathsf{T})$ of sheaves valued in $\mathsf{CAlg}(\mathsf{C})$ on T (Definition 1.8.7). Its object $\mathcal{R} \in \mathsf{Shv}_{\mathsf{CAlg}(\mathsf{C})}(\mathsf{T})$ is called a *sheaf of commutative ring objects* on T.

Definition 3.2.2. Let C be a symmetric monoidal ∞-category.

(1) A *ringed ∞-topos* is a pair $(\mathsf{T}, \mathcal{R})$ of an ∞-topos T and a sheaf of commutative algebras $\mathcal{R} \in \mathsf{Shv}_{\mathsf{CAlg}(\mathsf{C})}(\mathsf{T})$.

(2) A *functor* $(\mathsf{T}, \mathcal{R}) \to (\mathsf{T}', \mathcal{R}')$ of ringed ∞-topoi is a pair $(f, f^\sharp)$ of
- A geometric morphism $f : \mathsf{T} \to \mathsf{T}'$ of ∞-topoi corresponding to an adjunction $f^* : \mathsf{T}' \rightleftarrows \mathsf{T} : f_*$ (Definition 3.1.1).
- A morphism $f^\sharp : \mathcal{R}' \to f_*\mathcal{A}$ in $\mathsf{Shv}_{\mathsf{CAlg}(\mathsf{C})}(\mathsf{T}')$, where $f_*\mathcal{R}$ is defined by $(f_*\mathcal{R})(U') := \mathcal{R}(f^*U')$ for each $U' \in \mathsf{T}'$.

Our definition is an ∞-theoretical analogue of the ordinary ringed topos [SGA4, IV.11, 13]. Note that in (2) we used geometric morphisms (Definition 3.1.1), which is the correct notion of functors of ∞-topoi.

In the later sections we mainly discuss constructible sheaves by setting $k = \mathbb{F}_l$ or $\overline{\mathbb{Q}_\ell}$ and $\mathsf{C} := \mathsf{Mod}_k = \mathsf{N}(\mathrm{Mod}_k)$, the nerve of the category of k-modules. The monoidal structure on C is given by the standard $\otimes_k$.

An ethical remark is in order.

Remark. As mentioned in Remark 2.2.3, Lurie explores the theory of spectral algebraic geometry based on the sheaves of spectral rings in [Lur5]–[Lur14]. Under the notation above, his theory is developed in $\mathsf{Shv}_{\mathsf{CAlg}_\mathbb{K}}(\mathsf{T})$, where $\mathbb{K}$ denotes an $\mathbb{E}_\infty$-ring (see Appendix E.1) and $\mathsf{CAlg}_\mathbb{K} = \mathsf{CAlg}_\mathbb{K}(\mathsf{Mod}(\mathsf{Sp}))$ denotes the ∞-category of $\mathbb{K}$-algebra objects in $\mathsf{Mod}(\mathsf{Sp})$, which is the ∞-category of module objects in the ∞-category Sp of spectra equipped with the smash product monoidal structure. A pair $(\mathsf{T}, \mathcal{A})$ with $\mathcal{A} \in \mathsf{Shv}_{\mathsf{CAlg}_\mathbb{K}}(\mathsf{T})$ is called a *spectrally ringed ∞-topos* [Lur7, Definition 1.27].

As noted in [Lur7, Remark 2.1], a commutative rings is a discrete $\mathbb{E}_\infty$-ring, so that the theory for $\mathsf{C} = \mathsf{Mod}_k$ with k a commutative ring can be embedded in that for $\mathsf{C} = \mathsf{Mod}_\mathbb{K}(\mathsf{Sp})$ with $\mathbb{K}$ an arbitrary $\mathbb{E}_\infty$-ring. Thus our presentation is basically included in Lurie's exposition.

3.2.2. *Sheaves of $\mathcal{R}$-modules on ∞-topoi.* Let us first recall the notion of modules over commutative rings in the ∞-categorical setting. Let C be a symmetric monoidal ∞-category as in the previous part. Take a commutative algebra object $R \in \mathsf{CAlg}(\mathsf{C})$. Then we have the notion of *R-module objects* in C, and they form an ∞-category $\mathsf{Mod}_R(\mathsf{C})$. It is equipped with a symmetric monoidal structure, and the associated tensor product is denoted by $\otimes_R$.

Remark. Let us continue Remark 3.2.1. Since the commutative operad $\mathbb{E}_\infty^\otimes$ is coherent, we have the ∞-operad $\mathsf{Mod}_R^{\mathbb{E}_\infty}(\mathsf{C})^\otimes$ of R-module objects in C [Lur2, §3.3]. It is equipped with a fibration $\mathsf{Mod}_R^{\mathbb{E}_\infty}(\mathsf{C})^\otimes \to \mathbb{E}_\infty^\otimes$ of ∞-operads. We denoted $\mathsf{Mod}_R(\mathsf{C}) := \mathsf{Mod}_R^{\mathbb{E}_\infty}(\mathsf{C})$ above.

Now we turn to the discussion of sheaves on ∞-topoi. Let T be an ∞-topos and assume that the symmetric monoidal ∞-category C is presentable. Then by [Lur7, Lemma 1.13] the symmetric monoidal structure on C induces the *pointwise symmetric monoidal structure* on $\mathsf{Shv}_\mathsf{C}(\mathsf{T})$. In particular, we have the ∞-category $\mathsf{CAlg}(\mathsf{Shv}_\mathsf{C}(\mathsf{T}))$ of commutative ring objects in the symmetric monoidal ∞-category $\mathsf{Shv}_\mathsf{C}(\mathsf{T})$.

Let us continue to assume that C is presentable. As in the argument of [Lur7, Remark 1.18], the forgetful functor $\mathsf{CAlg}(\mathsf{C}) \to \mathsf{C}$ is conservative (Definition 1.3.5) and preserves small limits. Thus we have an equivalence

$$\mathsf{Shv}_{\mathsf{CAlg}(\mathsf{C})}(\mathsf{T}) \simeq \mathsf{CAlg}(\mathsf{Shv}_\mathsf{C}(\mathsf{T})),$$

where in the right hand side we are considering the pointwise symmetric monoidal structure.

Now let us take a sheaf $\mathcal{R} \in \mathsf{Shv}_{\mathsf{CAlg}(\mathsf{C})}(\mathsf{T})$ of commutative ring objects. Considering it as an object of $\mathsf{CAlg}(\mathsf{Shv}_\mathsf{C}(\mathsf{T}))$, we can consider the ∞-category $\mathsf{Mod}_\mathcal{R}(\mathsf{Shv}_\mathsf{C}(\mathsf{T}))$ of $\mathcal{R}$-module objects in the symmetric monoidal ∞-category $\mathsf{Shv}_\mathsf{C}(\mathsf{T})$. It is equipped with a symmetric monoidal structure. Moreover, by [Lur2, Theorem 3.4.4.2], the ∞-category $\mathsf{Mod}_\mathcal{R}(\mathsf{Shv}_\mathsf{C}(\mathsf{T}))$ is presentable, and the tensor product associated to the symmetric monoidal structure preserves small colimits in each variable. In total, we have

Proposition 3.2.3. Let C be a symmetric monoidal presentable ∞-category and T be an ∞-topos. For an $\mathcal{R} \in \mathsf{Shv}_{\mathsf{CAlg}(\mathsf{C})}(\mathsf{T})$, we have a symmetric monoidal presentable ∞-category

$$\mathsf{Mod}_{\mathcal{R}}(\mathsf{Shv}_{\mathsf{C}}(\mathsf{T}))$$

of $\mathcal{R}$-module objects. We call its object a *sheaf of* $\mathcal{R}$*-modules on* T. The associated tensor product

$$\otimes_{\mathcal{R}} : \mathsf{Mod}_{\mathcal{R}}(\mathsf{Shv}_{\mathsf{C}}(\mathsf{T})) \times \mathsf{Mod}_{\mathcal{R}}(\mathsf{Shv}_{\mathsf{C}}(\mathsf{T})) \longrightarrow \mathsf{Mod}_{\mathcal{R}}(\mathsf{Shv}_{\mathsf{C}}(\mathsf{T}))$$

preserves small colimits in each variable.

3.3. **Sheaves of spectra.** In this subsection we discuss sheaves of spectra following [Lur7]. We will use the theory of spectra in the ∞-categorical setting, which is developed extensively in [Lur2]. See also Appendix E where we give a brief summary.

3.3.1. *Sheaves of spectra in general setting.* Let C be an ∞-category admitting finite limits. We denote by

$$\mathsf{Sp}(\mathsf{C})$$

the ∞-category of spectra in C (Definition E.1.1). Hereafter C is assumed to be presentable (Definition B.8.3). Then the ∞-category $\mathsf{Sp}(\mathsf{C})$ is stable (Definition D.1.2) since a presentable ∞-category admits finite limits so that Fact E.1.4 works. Moreover $\mathsf{Sp}(\mathsf{C})$ has a t-structure (Definition D.2.2) by Fact E.1.5. Hereafter we call it *the* t-structure of $\mathsf{Sp}(\mathsf{C})$.

Let T be an ∞-topos. We now consider the ∞-category

$$\mathsf{Shv}_{\mathsf{Sp}(\mathsf{C})}(\mathsf{T})$$

of sheaves on T valued in spectra in C. This ∞-category inherits properties of $\mathsf{Sp}(\mathsf{C})$ mentioned above.

Lemma 3.3.1. Let C be a presentable ∞-category and T be an ∞-topos. Then the ∞-category $\mathsf{Shv}_{\mathsf{Sp}(\mathsf{C})}(\mathsf{T})$ is stable.

Proof. We follow the argument in [Lur7, Remark 1.3]. Since $\mathsf{Sp}(\mathsf{C})$ is stable, the ∞-category $\mathsf{Fun}(\mathsf{T}^{\mathrm{op}}, \mathsf{Sp}(\mathsf{C}))$ is also stable by [Lur2, Proposition 1.1.3.1]. Clearly $\mathsf{Shv}_{\mathsf{Sp}(\mathsf{C})}(\mathsf{T}) \subset \mathsf{Fun}(\mathsf{T}^{\mathrm{op}}, \mathsf{Sp}(\mathsf{C}))$ is closed under limits and translation. Then by [Lur2, Lemma 1.1.3.3] we have the conclusion. □

Lemma 3.3.2. Assume C is a presentable ∞-category equipped with a functor $\varepsilon : \mathsf{C} \to \mathcal{S}_*$ which preserves small limits. Here $\mathcal{S}_*$ denotes the ∞-category of pointed spaces (Definition E.1). Then the stable ∞-category $\mathsf{Shv}_{\mathsf{Sp}(\mathsf{C})}(\mathsf{T})$ has a t-structure determined by

$$\left(\mathsf{Shv}_{\mathsf{Sp}(\mathsf{C})}(\mathsf{T})_{\le 0}, \mathsf{Shv}_{\mathsf{Sp}(\mathsf{C})}(\mathsf{T})_{\ge 0}\right),$$

which will be given in Definition 3.3.5.

Our construction of t-structure is a slight modification of [Lur7, Proposition 1.7]. For the explanation, we need some preliminaries.

Let $\Omega^\infty : \mathsf{Sp}(\mathsf{C}) \to \mathsf{C}$ be the functor in Definition E.1.3. Composition with $\varepsilon : \mathsf{C} \to \mathcal{S}_*$ and the forgetful functor $\mathcal{S}_* \to \mathcal{S}$ yields $\mathsf{Sp}(\mathsf{C}) \to \mathcal{S}_*$, which further induces

$$\widetilde{\varepsilon} : \mathsf{Shv}_{\mathsf{Sp}(\mathsf{C})}(\mathsf{T}) \longrightarrow \mathsf{Shv}_{\mathcal{S}}(\mathsf{T})$$

since ε preserves limits. Now recall the equivalence $\mathsf{Shv}_{\mathcal{S}}(\mathsf{T}) \simeq \mathsf{T}$ (Fact 1.8.8). We also have the truncation functor $\tau_{\le 0} : \mathsf{T} \to \tau_{\le 0}\mathsf{T}$ (Definition B.9.4). Combining these functors, we have

Definition 3.3.3. (1) We define $\pi_0 : \mathsf{Shv}_{\mathsf{Sp}(\mathsf{C})}(\mathsf{T}) \to \tau_{\le 0}\mathsf{T}$ by

$$\pi_0 : \mathsf{Shv}_{\mathsf{Sp}(\mathsf{C})}(\mathsf{T}) \xrightarrow{\widetilde{\varepsilon}} \mathsf{Shv}_{\mathcal{S}}(\mathsf{T}) \xrightarrow{\sim} \mathsf{T} \xrightarrow{\tau_{\le 0}} \tau_{\le 0}\mathsf{T}.$$

(2) For $n \in \mathbb{Z}$, we define $\pi_n : \mathsf{Shv}_{\mathsf{Sp}(\mathsf{C})}(\mathsf{T}) \to \tau_{\le 0}\mathsf{T}$ by

$$\pi_n : \mathsf{Shv}_{\mathsf{Sp}(\mathsf{C})}(\mathsf{T}) \xrightarrow{\Omega^n} \mathsf{Shv}_{\mathsf{Sp}(\mathsf{C})}(\mathsf{T}) \xrightarrow{\pi_0} \tau_{\le 0}\mathsf{T}.$$

Here Ω^n is the functor induced by the iterated loop functor $\Omega^n : \mathsf{Sp}(\mathsf{C}) \to \mathsf{Sp}(\mathsf{C})$ (Definition D.1.4). We call the image $\pi_n\mathcal{M}$ of $\mathcal{M} \in \mathsf{Shv}_{\mathsf{Sp}(\mathsf{C})}(\mathsf{T})$ the *n-th homotopy group* of $\mathcal{M}$.

Remark. As explained in [Lur7, Remark 1.5], for $n \ge 2$, π_n can be regarded as a functor from $\mathrm{h}\,\mathsf{Shv}_{\mathsf{Sp}(\mathsf{C})}(\mathsf{T})$ to the category of abelian group objects in $\tau_{\le 0}\mathsf{T}$. In fact, π_n can be rewritten as the composition

$$\mathsf{Shv}_{\mathsf{Sp}(\mathsf{C})}(\mathsf{T}) \xrightarrow{\Omega^{n-2}} \mathsf{Shv}_{\mathsf{Sp}(\mathsf{C})}(\mathsf{T}) \longrightarrow \mathsf{Shv}_{\mathcal{S}_*}(\mathsf{T}) \xrightarrow{\sim} \mathsf{T}_* \xrightarrow{\pi_2} \tau_{\le 0}\mathsf{T},$$

where we used

- For an ∞-category B, the symbol B_* denotes the full sub-∞-category of $\mathsf{Fun}(\Delta^1, \mathsf{B})$ spanned by pointed objects (Definition E.1).
- The functor $\mathsf{Shv}_{\mathsf{Sp}(\mathsf{C})}(\mathsf{T}) \to \mathsf{Shv}_{\mathcal{S}_*}(\mathsf{T})$ is given by the composition $\mathsf{Shv}_{\mathsf{Sp}(\mathsf{C})}(\mathsf{T}) \to \mathsf{Shv}_{\mathsf{Sp}(\mathcal{S}_*)}(\mathsf{T}) \to \mathsf{Shv}_{\mathcal{S}_*}(\mathsf{T})$. The first functor is induced by the limit-preserving $\varepsilon : \mathsf{C} \to \mathcal{S}_*$. The second one comes from the description of $\mathsf{Sp}(\mathcal{S}_*)$ as the limit of the tower

$$\cdots \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_*.$$

- The equivalence $\mathsf{Shv}_{\mathcal{S}_*}(\mathsf{T}) \simeq \mathsf{T}_*$ is shown similarly to $\mathsf{Shv}_{\mathcal{S}}(\mathsf{T}) \simeq \mathsf{T}$ (Fact 1.8.8).

Then the image of π_n is included in the full sub-∞-category $\mathcal{EM}_n(\tau_{\le 0}\mathsf{T}) \subset (\tau_{\le 0}\mathsf{T})_*$ of Eilenberg-MacLane objects [Lur1, Definition 7.2.2.1]. In the case $\mathsf{C} = \mathcal{S}_*$ and $\varepsilon = \mathrm{id}$, then the image of π_n is equal to $\mathcal{EM}_n(\tau_{\le 0}\mathsf{T})$. For $n \ge 2$, the ∞-category $\mathcal{EM}_n(\tau_{\le 0}\mathsf{T})$ is equivalent to the ∞-category of abelian group objects by [Lur1, Proposition 7.2.2.12].

3.3.2. *Sheaves of spectra.* We now take $\mathsf{C} = \mathcal{S}_*$ and $\varepsilon = \mathrm{id}$ in the argument so far, and consider the ∞-category $\mathsf{Sp} = \mathsf{Sp}(\mathcal{S}_*)$ of spectra. See Appendix E.1 for a summary. Let us mention here that the *smash product* gives a symmetric monoidal structure on $\mathcal{S}$ [Lur2, §6.3.2].

By the argument in the previous §3.3.1, we have the ∞-category

$$\mathsf{Shv}_{\mathsf{Sp}}(\mathsf{T})$$

of sheaves of spectra on an ∞-topos T. We list its formal properties below.

Lemma 3.3.4. (1) $\mathsf{Shv}_{\mathsf{Sp}}(\mathsf{T})$ is stable and equipped with a t-structure $(\mathsf{Shv}_{\mathsf{Sp}}(\mathsf{T})_{\le 0}, \mathsf{Shv}_{\mathsf{Sp}}(\mathsf{T})_{\ge 0})$ given in Lemma 3.3.2. Hereafter we call it *the* t-structure of $\mathsf{Shv}_{\mathsf{Sp}}(\mathsf{T})$.
(2) $\mathsf{Shv}_{\mathsf{Sp}}(\mathsf{T})$ has a symmetric monoidal structure induced by the smash product on Sp. Hereafter we call it the *smash product symmetric monoidal structure.* Moreover it is compatible with the t-structure in the sense of Definition D.2.6.
(3) $\mathsf{Shv}_{\mathsf{Sp}}(\mathsf{T})$ is presentable.

Proof. (1) is discussed in the previous §3.3.1. (3) is stated in [Lur5, Remark 1.1.5]. (2) is discussed in [Lur7, §1] with much length, so let us give a summary. The symmetric monoidal structure on Sp given by the smash product induces a symmetric monoidal structure on the ∞-category $\mathsf{Fun}(K, \mathsf{Sp})$ of morphisms for any simplicial set K by [Lur2, Remark 2.1.3.4]. Then by [Lur8, Proposition 1.15], it further induces a symmetric monoidal structure on $\mathsf{Shv}_{\mathsf{Sp}}(\mathsf{T})$. Also it is compatible with the t-structure by [Lur7, Proposition 1.16]. □

For later use, let us introduce

Definition 3.3.5 ([Lur7, Definition 1.6])**.** Let T an ∞-topos and $\mathcal{M} \in \mathsf{Shv}_{\mathsf{Sp}}(\mathsf{T})$.
(1) $\mathcal{M}$ is *connective* if the homotopy groups $\pi_n\mathcal{M}$ vanish for $n \in \mathbb{Z}_{<0}$. We denote by $\mathsf{Shv}_{\mathsf{Sp}}(\mathsf{T})_{\ge 0}$ the full sub-∞-category spanned by connective objects.
(2) $\mathcal{M}$ is *coconnective* if $\Omega^\infty M$ is a discrete object, i.e., a 0-truncated object (Definition B.9.1). We denote by $\mathsf{Shv}_{\mathsf{Sp}}(\mathsf{T})_{\le 0}$ the full sub-∞-category spanned by coconnective objects.

3.4. **Sheaves of commutative ring spectra and modules.** In this subsection we discuss sheaves of stable $\mathcal{A}$-modules with $\mathcal{A}$ a sheaf of commutative ring spectra following [Lur8, §2.1]. As mentioned in the beginning of this section, these objects can be regarded as ∞-theoretic counterparts of complexes of sheaves, and the corresponding ∞-categories are counterparts of derived categories.

3.4.1. *Sheaves of commutative ring spectra.* Let $\mathsf{CAlg}(\mathsf{Sp})$ be the ∞-category of commutative ring objects in Sp regarded as a symmetric monoidal category with respect to smash product monoidal structure. An object of CAlg is nothing but an $\mathbb{E}_\infty$-ring (Appendix E.2), so we call $\mathsf{CAlg}(\mathsf{Sp})$ the *∞-category of $\mathbb{E}_\infty$-rings.*

Let T be an ∞-topos. Since $\mathsf{Shv}_{\mathsf{Sp}}(\mathsf{T})$ is a symmetric monoidal presentable ∞-category by Lemma 3.3.4, we can apply the arguments in §3.2.1 to $\mathsf{C} = \mathsf{Shv}_{\mathsf{Sp}}(\mathsf{T})$. Thus we have the ∞-category

$$\mathsf{CAlg}(\mathsf{Shv}_{\mathsf{Sp}}(\mathsf{T}))$$

of commutative ring objects. As mentioned in [Lur7, Remark 1.18], since $\mathsf{CAlg}(\mathsf{Sp}) \to \mathsf{Sp}$ is conservative (Definition 1.3.5) and preserves small limits, we have an equivalence

$$\mathsf{CAlg}(\mathsf{Shv}_{\mathsf{Sp}}(\mathsf{T})) \simeq \mathsf{Shv}_{\mathsf{CAlg}(\mathsf{Sp})}(\mathsf{T}).$$

Thus we may call an object of $\mathsf{CAlg}(\mathsf{Shv}_{\mathsf{Sp}}(\mathsf{T}))$ a *sheaf of commutative ring spectra on* T.

3.4.2. *Sheaves of stable modules over commutative ring spectra.* Let us continue to use the symbols in the previous part. Next we apply the argument in §3.2.2 to the symmetric monoidal presentable ∞-category $\mathsf{C} = \mathsf{Shv}_{\mathsf{Sp}}(\mathsf{T})$.

Definition. Let T be an ∞-topos, and $\mathcal{R} \in \mathsf{Shv}_{\mathsf{CAlg}(\mathsf{Sp})}(\mathsf{T})$ be a sheaf of commutative ring spectra. We denote the ∞-category of $\mathcal{R}$-module objects in $\mathsf{Shv}_{\mathsf{Sp}}(\mathsf{T})$ by

$$\mathsf{Mod}^{\mathrm{stab}}_{\mathcal{R}}(\mathsf{T}) := \mathsf{Mod}_{\mathcal{R}}(\mathsf{Shv}_{\mathsf{Sp}}(\mathsf{T}))$$

and call it the ∞-category of *stable $\mathcal{R}$-modules* on T.

The ∞-category $\mathsf{Mod}^{\mathrm{stab}}_{\mathcal{R}}(\mathsf{T})$ has a symmetric monoidal structure whose tensor product is denoted by

$$- \otimes_{\mathcal{R}} - : \mathsf{Mod}^{\mathrm{stab}}_{\mathcal{R}}(\mathsf{T}) \times \mathsf{Mod}^{\mathrm{stab}}_{\mathcal{R}}(\mathsf{T}) \longrightarrow \mathsf{Mod}^{\mathrm{stab}}_{\mathcal{R}}(\mathsf{T}).$$

We cite some formal properties of $\mathsf{Mod}^{\mathrm{stab}}_{\mathcal{R}}(\mathsf{T})$.

Fact 3.4.1 ([Lur8, Proposition 2.1.3])**.** Let T be an ∞-topos and $\mathcal{R} \in \mathsf{Shv}_{\mathsf{CAlg}(\mathsf{Sp})}(\mathsf{T})$.

(1) The ∞-category $\mathsf{Mod}^{\mathrm{stab}}_{\mathcal{R}}(\mathsf{T})$ is stable.
(2) The ∞-category $\mathsf{Mod}^{\mathrm{stab}}_{\mathcal{R}}(\mathsf{T})$ is presentable and the tensor product $\otimes_{\mathcal{R}}$ preserves small colimits in each variable.
(3) The forgetful functor $\theta : \mathsf{Mod}^{\mathrm{stab}}_{\mathcal{R}}(\mathsf{T}) \to \mathsf{Shv}_{\mathsf{Sp}}(\mathsf{T})$ is conservative (Definition 1.3.5) and preserves small limits and colimits.

The item (1) is the origin of our naming of *stable* $\mathcal{R}$-modules.

Let us next recall the natural t-structure on $\mathsf{Mod}^{\mathrm{stab}}_{\mathcal{R}}(\mathsf{T})$ which is induced by the t-structure on $\mathsf{Shv}_{\mathsf{Sp}}(\mathsf{T})$ in Lemma 3.3.2. Recall the ∞-categories $\mathsf{Shv}_{\mathsf{Sp}}(\mathsf{T})_{\geq 0}$ and $\mathsf{Shv}_{\mathsf{Sp}}(\mathsf{T})_{\leq 0}$ in Definition 3.3.5.

Fact 3.4.2 ([Lur8, Proposition 2.1.3])**.** Let T be an ∞-topos and $\mathcal{R} \in \mathsf{Shv}_{\mathsf{CAlg}(\mathsf{Sp})}(\mathsf{T})$. Assume that $\mathcal{R}$ is connective (Definition 3.3.5).

(1) The stable ∞-category $\mathsf{Mod}^{\mathrm{stab}}_{\mathcal{R}}(\mathsf{T})$ has a t-structure determined by
$$\mathsf{Mod}^{\mathrm{stab}}_{\mathcal{R}}(\mathsf{T})_{\geq 0} := \theta^{-1}\,\mathsf{Shv}_{\mathsf{Sp}}(\mathsf{T})_{\geq 0}, \quad \mathsf{Mod}^{\mathrm{stab}}_{\mathcal{R}}(\mathsf{T})_{\leq 0} := \theta^{-1}\,\mathsf{Shv}_{\mathsf{Sp}}(\mathsf{T})_{\leq 0}.$$
Here $\theta : \mathsf{Mod}^{\mathrm{stab}}_{\mathcal{R}}(\mathsf{T}) \to \mathsf{Shv}_{\mathsf{Sp}}(\mathsf{T})$ denotes the forgetful functor. We call it *the* t-structure of $\mathsf{Mod}^{\mathrm{stab}}_{\mathcal{R}}(\mathsf{T})$ from now.
(2) The t-structure on $\mathsf{Mod}_{\mathcal{R}}(\mathsf{T})$ is accessible, that is, the ∞-category $\mathsf{Mod}_{\mathcal{R}}(\mathsf{T})_{\geq 0}$ is presentable.
(3) The t-structure on $\mathsf{Mod}^{\mathrm{stab}}_{\mathcal{R}}(\mathsf{T})$ is compatible with the symmetric monoidal structure. In other words, the sub-∞-category $\mathsf{Mod}^{\mathrm{stab}}_{\mathcal{R}}(\mathsf{T})_{\geq 0}$ contains the unit object of $\mathsf{Mod}^{\mathrm{stab}}_{\mathcal{R}}(\mathsf{T})$ is stable under tensor product.
(4) The t-structure on $\mathsf{Mod}^{\mathrm{stab}}_{\mathcal{R}}(\mathsf{T})$ is compatible with filtered colimits. In other words, the sub-∞-category $\mathsf{Mod}^{\mathrm{stab}}_{\mathcal{R}}(\mathsf{T})_{\leq 0}$ is stable under filtered colimits.
(5) The t-structure on $\mathsf{Mod}_{\mathcal{R}}(\mathsf{T})$ is right complete (Definition D.2.5).

For $\mathcal{R} \in \mathsf{Shv}_{\mathsf{CAlg}}(\mathsf{T})$, the 0-th homotopy $\pi_0\mathcal{R}$ (Definition 3.3.3) is a commutative ring object of $\mathsf{Shv}_{\mathcal{S}}(\mathsf{T}) \simeq \mathsf{T}$, where the equivalence is given by Fact 1.8.8. Let us denote by $\mathsf{Mod}_{\pi_0\mathcal{R}}(\mathsf{Shv}_{\mathcal{S}}(\mathsf{T}))$ the ∞-category of $\pi_0\mathcal{R}$-module objects. Then the equivalence $\mathsf{Shv}_{\mathcal{S}}(\mathsf{T}) \simeq \mathsf{T}$ induces $\mathsf{Mod}_{\pi_0\mathcal{R}}(\mathsf{Shv}_{\mathcal{S}}(\mathsf{T})) \simeq \mathsf{Mod}_{\pi_0\mathcal{R}}(\mathsf{T})$. If we further assume $\mathcal{R}$ is connected, then the following statement holds. The proof is by definition and omitted.

Lemma 3.4.3 ([Lur8, Remark 2.1.5])**.** For a connective sheaf $\mathcal{R}$ of $\mathbb{E}_\infty$-rings on an ∞-topos T, we have

$$\mathsf{Mod}^{\mathrm{stab}}_{\mathcal{R}}(\mathsf{T})^{\heartsuit} \simeq \mathsf{Mod}_{\pi_0\mathcal{R}}(\mathsf{T}),$$

where the left hand side is the heart (Definition D.2.4) of the t-structure of Fact 3.4.2 (1).

For later use, let us introduce

Notation ([Lur2, Notation 1.1.2.17])**.** Let C be a stale ∞-category, and let $X, Y \in \mathsf{C}$. We define the abelian group $\mathrm{Ext}^n_{\mathsf{C}}(X, Y)$ by

$$\mathrm{Ext}^n_{\mathsf{C}}(X, Y) := \mathrm{Hom}_{\mathrm{h}\,\mathsf{C}}(X[-n], Y).$$

where $[-n]$ denotes the shift (Definition D.1.4) in the stable ∞-category C. For $n = 0$, we denote

$$\mathrm{Hom}_{\mathsf{C}}(X, Y) := \mathrm{Ext}^0_{\mathsf{C}}(X, Y) = \mathrm{Hom}_{\mathrm{h}\,\mathsf{C}}(X, Y).$$

For $n \in \mathbb{Z}_{\geq 0}$, we can identify $\mathrm{Ext}^n_{\mathsf{C}}(X, Y) \simeq \pi_{-n}\,\mathrm{Map}_{\mathsf{C}}(X, Y)$.

Notation. Let T be an ∞-topos and $\mathcal{R} \in \mathsf{Shv}_{\mathsf{CAlg}(\mathsf{Sp})}(\mathsf{T})$. For $\mathcal{M}, \mathcal{N} \in \mathsf{Mod}^{\mathrm{stab}}_{\mathcal{R}}(\mathsf{T})$ and $n \in \mathbb{Z}$, we denote

$$\mathrm{Ext}^n_{\mathcal{R}}(\mathcal{M}, \mathcal{N}) := \mathrm{Ext}^n_{\mathsf{Mod}^{\mathrm{stab}}_{\mathcal{R}}(\mathsf{T})}(\mathcal{M}, \mathcal{N}), \quad \mathrm{Hom}_{\mathcal{R}}(\mathcal{M}, \mathcal{N}) := \mathrm{Ext}^0_{\mathcal{R}}(\mathcal{M}, \mathcal{N}).$$

3.4.3. *Functors on stable $\mathcal{R}$-modules.* Fix an ∞-topos T and take $\mathcal{R} \in \mathsf{Shv}_{\mathsf{CAlg(Sp)}}(\mathsf{T})$. In this part we introduce some functors on the stable ∞-category of stable $\mathcal{R}$-modules.

Let us begin with internal Hom functor, which will give an ∞-theoretical counterpart of the tensor-hom adjunction. For $\mathcal{M} \in \mathsf{Mod}^{\mathrm{stab}}_{\mathcal{R}}(\mathsf{T})$, the functor

$$- \otimes_{\mathcal{R}} \mathcal{M} : \mathsf{Mod}^{\mathrm{stab}}_{\mathcal{R}}(\mathsf{T}) \longrightarrow \mathsf{Mod}^{\mathrm{stab}}_{\mathcal{R}}(\mathsf{T})$$

is right exact (Definition B.10.1) since it preserves small colimits by Fact 3.4.1 (2) and we can apply the criterion of right exactness (Fact B.10.2 (2)). Then by Fact B.10.3 there is a right adjoint of $- \otimes_{\mathcal{R}} \mathcal{M}$. We denote it by

$$\mathscr{H}om_{\mathcal{R}}(\mathcal{M}, -) : \mathsf{Mod}^{\mathrm{stab}}_{\mathcal{R}}(\mathsf{T}) \longrightarrow \mathsf{Mod}^{\mathrm{stab}}_{\mathcal{R}}(\mathsf{T}).$$

By a similar argument on the first variable, we obtain a bifunctor

$$\mathscr{H}om_{\mathcal{R}}(-, -) : \mathsf{Mod}^{\mathrm{stab}}_{\mathcal{R}}(\mathsf{T})^{\mathrm{op}} \times \mathsf{Mod}^{\mathrm{stab}}_{\mathcal{R}}(\mathsf{T}) \longrightarrow \mathsf{Mod}^{\mathrm{stab}}_{\mathcal{R}}(\mathsf{T}).$$

We call it the *internal Hom functor*.

The internal Hom functor agrees with the *morphism object* [Lur2, Definition 4.2.1.28]. Then by [Lur2, Proposition 4.2.1.33] we have

Lemma. For any $\mathcal{L}, \mathcal{M}, \mathcal{N} \in \mathsf{Mod}^{\mathrm{stab}}_{\mathcal{R}}(\mathsf{T})$, there is a functorial equivalence

$$\mathscr{H}om_{\mathcal{R}}(\mathcal{L} \otimes_{\mathcal{R}} \mathcal{M}, \mathcal{N}) \xrightarrow{\sim} \mathscr{H}om_{\mathcal{R}}(\mathcal{L}, \mathscr{H}om_{\mathcal{R}}(\mathcal{M}, \mathcal{N}))$$

which is unique up to contractible ambiguity.

Next we introduce the direct image and inverse image functors. Our presentation basically follows [Lur8, §2.5] but with a slight modification. Let

$$(f, f^{\sharp}) : (\mathsf{T}, \mathcal{R}) \longrightarrow (\mathsf{T}', \mathcal{R}')$$

be a morphism of ringed ∞-topoi (Definition 3.2.2). Thus $f : \mathsf{T} \to \mathsf{T}'$ is a geometric morphism of ∞-topoi with the associated adjunction $f^* : \mathsf{T}' \rightleftarrows \mathsf{T} : f_*$, and $f^{\sharp} : \mathcal{R}' \to f_*\mathcal{R}$ is a morphism in $\mathsf{Shv}_{\mathsf{CAlg(Sp)}}(\mathsf{T}')$.

Composition with $f_* : \mathsf{T} \to \mathsf{T}'$ gives a symmetric monoidal functor $f^{-1} : \mathsf{Shv}_{\mathsf{Sp}}(\mathsf{T}') \longrightarrow \mathsf{Shv}_{\mathsf{Sp}}(\mathsf{T})$. Then we can regard $f^{-1}\mathcal{R}'$ as an object of $\mathsf{Shv}_{\mathsf{CAlg(Sp)}}(\mathsf{T})$. Moreover f^{-1} induces a functor $\mathsf{Mod}^{\mathrm{stab}}_{\mathcal{R}'}(\mathsf{T}') \to \mathsf{Mod}^{\mathrm{stab}}_{f^{-1}\mathcal{R}'}(\mathsf{T})$. Abusing the symbol, we denote it by

$$f^{-1} : \mathsf{Mod}^{\mathrm{stab}}_{\mathcal{R}'}(\mathsf{T}') \longrightarrow \mathsf{Mod}^{\mathrm{stab}}_{f^{-1}\mathcal{R}'}(\mathsf{T}).$$

Note also that $f^{\sharp}$ and f yield a morphism $f^{-1}\mathcal{R}' \to \mathcal{R}$ in $\mathsf{Shv}_{\mathsf{CAlg}_{\mathbb{K}}}(\mathsf{T})$ so that we can regard $\mathcal{R}$ as an object in $\mathsf{Mod}^{\mathrm{stab}}_{f^{-1}\mathcal{R}'}(\mathsf{T})$. Thus we have the functor

$$f^* : \mathsf{Mod}^{\mathrm{stab}}_{\mathcal{R}'}(\mathsf{T}') \longrightarrow \mathsf{Mod}^{\mathrm{stab}}_{\mathcal{R}}(\mathsf{T}), \quad \mathcal{M}' \longmapsto f^*\mathcal{M}' := f^{-1}\mathcal{M}' \otimes_{f^{-1}\mathcal{R}'} \mathcal{R}.$$

Since the inverse image functor $f^* : \mathsf{Mod}^{\mathrm{stab}}_{\mathcal{R}'}(\mathsf{T}') \to \mathsf{Mod}^{\mathrm{stab}}_{\mathcal{R}}(\mathsf{T})$ preserves colimits, and since the ∞-categories of stable $\mathcal{R}$-modules are presentable (Fact 3.4.1 (2)), there is a right adjoint

$$f_* : \mathsf{Mod}^{\mathrm{stab}}_{\mathcal{R}}(\mathsf{T}) \longrightarrow \mathsf{Mod}^{\mathrm{stab}}_{\mathcal{R}'}(\mathsf{T}').$$

It is also described as follows. The composition with $f^* : \mathsf{T}' \to \mathsf{T}$ gives a functor $\mathsf{Shv}_{\mathsf{Sp}}(\mathsf{T}) \longrightarrow \mathsf{Shv}_{\mathsf{Sp}}(\mathsf{T}')$, which induces $\mathsf{Mod}^{\mathrm{stab}}_{\mathcal{R}}(\mathsf{T}) \to \mathsf{Mod}^{\mathrm{stab}}_{f_*\mathcal{R}}(\mathsf{T}')$. Then the morphism $f^{\sharp}$ induces $f_* : \mathsf{Mod}^{\mathrm{stab}}_{\mathcal{R}}(\mathsf{T}) \to \mathsf{Mod}^{\mathrm{stab}}_{\mathcal{R}'}(\mathsf{T}')$.

Notation 3.4.4. We call f^* the *inverse image functor*, and f_* the *direct image functor*.

3.5. **Sheaves of commutative rings and modules on ∞-topoi.** This subsection is a complement of the previous §3.4. We will give notations for sheaves of commutative rings, sheaves of modules over them, derived categories and functors between them on ∞-topoi. These will be essentially the same with those on ordinary topoi, and we write them down just for the completeness of our presentation.

3.5.1. *Sheaves of modules over commutative rings.* Let Ab denote the category of additive groups, and $\mathsf{Ab} := \mathsf{N}(\mathrm{Ab})$ be its nerve. The ∞-category Ab has the symmetric monoidal structure. Also we see that Ab is a presentable ∞-category. One can show it directly by definition, and also by using the equivalence [Lur1, Proposition A.3.7.6] between a presentable ∞-category and the nerve of the subcategory of fibrant-cofibrant objects in a combinatorial simplicial model category.

Let T be an ∞-topos. We can now apply the argument in §3.2.2 to the symmetric monoidal presentable ∞-category $\mathsf{C} = \mathsf{Ab}$. Noting that $\mathsf{CAlg(Ab)} \simeq \mathsf{Com}$ which is the ∞-category of commutative rings (§2.1), We call the ∞-category

$$\mathsf{CAlg}(\mathsf{Shv}_{\mathsf{Ab}}(\mathsf{T})) \simeq \mathsf{Shv}_{\mathsf{CAlg(Ab)}}(\mathsf{T}) \simeq \mathsf{Shv}_{\mathsf{Com}}(\mathsf{T})$$

the ∞-*category of sheaves of commutative rings on* T.

Taking a sheaf $\mathcal{A} \in \mathsf{Shv}_{\mathsf{Com}}(\mathsf{T})$ of commutative rings, we also have the ∞-category of *sheaves of* $\mathcal{A}$*-modules on* T (Proposition 3.2.3). We denote it by

$$\mathsf{Mod}_{\mathcal{A}}(\mathsf{T}) := \mathsf{Mod}_{\mathcal{A}}(\mathsf{Shv}_{\mathsf{Ab}}(\mathsf{T})).$$

Recall that it is a symmetric monoidal presentable ∞-category. The associated tensor product is denoted by $\otimes_{\mathcal{A}}$. We also have the following standard claim.

Proposition 3.5.1. The homotopy category $\mathrm{A} := \mathrm{h}\,\mathsf{Mod}_{\mathcal{A}}(\mathsf{T})$ is a Grothendieck abelian category (Definition D.3.3).

Proof. We omit the proof of the claim that A is an abelian category since the argument for ordinary topos works. The claim that A is a presentable category is a consequence of the presentability of the ∞-category $\mathsf{Mod}_{\mathcal{A}}(\mathsf{T})$.

Let us show that the class of monomorphisms is preserved by small filtered colimits Let $\{f_i : A_i \to B_i\}$ be a filtered diagram of monomorphisms in A. We have the corresponding filtered diagram of fiber sequences $A_i \xrightarrow{f_i} B_i \to B_i/A_i$ in $\mathsf{Mod}_{\mathcal{A}}(\mathsf{T})$. We can take the filtered colimit since $\mathsf{Mod}_{\mathcal{A}}(\mathsf{T})$ is presentable so that it has colimit. The resulting colimit $A \xrightarrow{f} B \to B/A$ is a fiber sequence, so that f is a monomorphism in A. □

Remark. The presentability of A means that there is a small family of generators. We can explicitly give such generators as follows. Recall from Fact 3.1.5 that we have the canonical functor $j_! : \mathsf{T}_{/U} \to \mathsf{T}$ for each $U \in \mathsf{T}$, and we have the biadjunction $j_! : \mathsf{T}_{/U} \rightleftarrows \mathsf{T} : j^*$ and $j^* : \mathsf{T} \rightleftarrows \mathsf{T}_{/U} : j_*$. Then, for $\mathcal{A} \in \mathsf{Shv}_{\mathsf{CAlg(Ab)}}(\mathsf{T})$, we have $\mathcal{A}|_U := j^*\mathcal{A} \in \mathsf{Shv}_{\mathsf{CAlg(Ab)}}(\mathsf{T}_{/U})$ (Definition 3.2.2, Notation 3.1.6). Then we also have an adjunction

$$j_! : \mathsf{Mod}_{\mathcal{A}|_U}(\mathsf{T}_{/U}) \rightleftarrows \mathsf{Mod}_{\mathcal{A}}(\mathsf{T}) : j^*$$

of ∞-categories. Now $\{j_! j^* \mathcal{A}\}_{U \in \mathsf{T}}$ gives the desired family of generators (note that we tacitly assume that T is small). In fact, for each $\mathcal{M} \in \mathsf{Mod}_{\mathcal{A}}(\mathsf{T})$ the adjunction gives

$$\mathrm{Map}_{\mathsf{Mod}_{\mathcal{A}}(\mathsf{T})}(j_! j^*\mathcal{A}, \mathcal{M}) \simeq \mathrm{Map}_{\mathsf{Mod}_{\mathcal{A}|_U}(\mathsf{T}_{/U})}(j^*\mathcal{A}, j^*\mathcal{M}) \simeq (j^*\mathcal{M})(U) = \mathcal{M}(U).$$

Now the same argument as §3.4.3 gives functors on sheaves of modules over sheaves of commutative rings. We list them in the following proposition.

Proposition. Let T, T' be ∞-topoi and $\mathcal{A}, \mathcal{A}'$ be sheaves of commutative rings on T, T' respectively. Let also $(f, f^\sharp) : (\mathsf{T}, \mathcal{A}) \longrightarrow (\mathsf{T}', \mathcal{A}')$ be a functor of ringed ∞-topoi.

(1) For $\mathcal{M} \in \mathsf{Mod}_{\mathcal{A}}(\mathsf{T})$, the right exact functor

$$- \otimes_{\mathcal{A}} \mathcal{M} : \mathsf{Mod}_{\mathcal{A}}(\mathsf{T}) \longrightarrow \mathsf{Mod}_{\mathcal{A}}(\mathsf{T})$$

has a right adjoint denoted by

$$\mathscr{H}om_{\mathcal{A}}(\mathcal{M}, -) : \mathsf{Mod}_{\mathcal{A}}(\mathsf{T}) \longrightarrow \mathsf{Mod}_{\mathcal{A}}(\mathsf{T}).$$

It gives rise to a bifunctor

$$\mathscr{H}om_{\mathcal{A}}(-, -) : \mathsf{Mod}_{\mathcal{A}}(\mathsf{T})^{\mathrm{op}} \times \mathsf{Mod}_{\mathcal{A}}(\mathsf{T}) \longrightarrow \mathsf{Mod}_{\mathcal{A}}(\mathsf{T})$$

called the *internal Hom functor*.

(2) We have the *inverse image functor*

$$f^* : \mathsf{Mod}_{\mathcal{A}'}(\mathsf{T}') \longrightarrow \mathsf{Mod}_{\mathcal{A}}(\mathsf{T}), \quad \mathcal{M}' \longmapsto f^*\mathcal{M}' := f^{-1}\mathcal{M}' \otimes_{f^{-1}\mathcal{A}'} \mathcal{A}.$$

It is right exact, and have a a right adjoint

$$f_* : \mathsf{Mod}_{\mathcal{A}}(\mathsf{T}) \longrightarrow \mathsf{Mod}_{\mathcal{A}'}(\mathsf{T}').$$

called the *direct image functor*.

3.5.2. *Derived ∞-categories.* For a Grothendieck abelian category A one can construct the *derived-∞-category* $\mathsf{D}_\infty(\mathrm{A})$, which is an ∞-categorical counterpart of the unbounded derived category of A. It is a stable ∞-category equipped with a t-structure, which is also an ∞-categorical counterpart of triangulated category with a t-structure. See Appendix D.3.2 for an account on $\mathsf{D}_\infty(\mathrm{A})$.

Let T be an ∞-topos and $\mathcal{A}$ be a sheaf of commutative rings on T. Denote by $\mathsf{Mod}_{\mathcal{A}}(\mathsf{T})$ the ∞-category of $\mathcal{A}$-modules on T. By Proposition 3.5.1,

$$\mathrm{Mod}_{\mathcal{A}}(\mathsf{T}) := \mathrm{h}\,\mathsf{Mod}_{\mathcal{A}}(\mathsf{T})$$

is a Grothendieck abelian category, so we can apply to it the construction in Appendix D.3.2. Thus we have the *derived ∞-category of sheaves of $\mathcal{A}$-modules on* T

$$\mathsf{D}_\infty(\mathrm{Mod}_{\mathcal{A}}(\mathsf{T})) = \mathsf{N}_{\mathrm{dg}}(\mathrm{C}(\mathrm{Mod}_{\mathcal{A}}(\mathsf{T}))^\circ).$$

We collect its properties in

Lemma 3.5.2. The derived ∞-category $\mathsf{D} := \mathsf{D}_\infty(\mathrm{Mod}_{\mathcal{A}}(\mathsf{T}))$ enjoys the following properties.
1. D is a stable ∞-category (Definition D.1.2).
2. D is equipped with a t-structure (Definition D.2.2) determined by $(\mathsf{D}_{\le 0}, \mathsf{D}_{\ge 0})$. Here $\mathsf{D}_{\le 0}$ (resp. $\mathsf{D}_{\ge 0}$) is the full sub-∞-category of D spanned by those objects $\mathcal{M}$ such that $H_n(\mathcal{M}) = 0$ for any $n > 0$ (resp. $n < 0$). Hereafter we call it *the* t-structure on D.
3. The heart of the t-structure (Definition D.2.4) is $\mathsf{D}^\heartsuit \simeq \mathsf{Mod}_{\mathcal{A}}(\mathsf{T})$.
4. D is right complete with respect to the t-structure (Definition D.2.5).
5. D is accessible with respect to the t-structure (Definition D.2.6).

Proof. All the claims are explained in Appendix D.3.2 and D.3.3. □

In particular, the homotopy category $\mathrm{h}\,\mathsf{D}_\infty(\mathsf{Mod}_{\mathcal{A}}(\mathsf{T}))$ is a triangulated category with a t-structure. It can be regarded as the unbounded derived category of the abelian category $\mathrm{Mod}_{\mathcal{A}}(\mathsf{T})$ as mentioned before.

Since a Grothendieck abelian category has enough injective objects, we also have the left bounded derived ∞-category (Definition D.3.2)

$$\mathsf{D}^+_\infty(\mathrm{Mod}_{\mathcal{A}}(\mathsf{T})) = \mathsf{N}_{\mathrm{dg}}(\mathrm{C}^+(\mathrm{Mod}_{\mathcal{A}}(\mathsf{T})_{\mathrm{inj}})).$$

It is stable and equipped with a t-structure. Moreover, by Fact D.3.8, we have a fully faithful t-exact functor (Definition D.3.9)

$$\mathsf{D}^+_\infty(\mathrm{Mod}_{\mathcal{A}}(\mathsf{T})) \longrightarrow \mathsf{D}_\infty(\mathrm{Mod}_{\mathcal{A}}(\mathsf{T}))$$

whose essential image is $\cup_{n\in\mathbb{Z}} \mathsf{D}_\infty(\mathrm{Mod}_{\mathcal{A}}(\mathsf{T}))_{\le n}$.

Recall that we have the stable ∞-category $\mathsf{Mod}^{\mathrm{stab}}_{\mathcal{A}}(\mathsf{T})$ of stable $\mathcal{A}$-modules by taking $\mathcal{R} := \mathcal{A}$ in the argument of §3.4.2. It is equipped with a right complete t-structure. Thus we also have a fully faithful t-exact functor

$$\mathsf{D}^+_\infty(\mathrm{Mod}_{\mathcal{A}}(\mathsf{T})) \longrightarrow \mathsf{Mod}^{\mathrm{stab}}_{\mathcal{A}}(\mathsf{T})$$

whose essential image is $\cup_{n\in\mathbb{Z}} \mathsf{Mod}^{\mathrm{stab}}_{\mathcal{A}}(\mathsf{T})_{\le n}$. Since $\mathsf{D}_\infty(\mathrm{Mod}_{\mathcal{A}}(\mathsf{T}))$ and $\mathrm{Mod}^{\mathrm{stab}}_{\mathcal{A}}(\mathsf{T})$ are both right complete, we have

Proposition 3.5.3. The fully faithful t-exact functor $\mathsf{D}_\infty(\mathrm{Mod}_{\mathcal{A}}(\mathsf{T})) \hookrightarrow \mathsf{Mod}^{\mathrm{stab}}_{\mathcal{A}}(\mathsf{T})$ yields an equivalence of ∞-categories

$$\mathsf{D}_\infty(\mathrm{Mod}_{\mathcal{A}}(\mathsf{T})) \simeq \mathsf{Mod}^{\mathrm{stab}}_{\mathcal{A}}(\mathsf{T}).$$

In other words, the ∞-category of $\mathcal{A}$-module spectra can be identified with the derived ∞-category of the abelian category of (discrete) $\mathcal{A}$-modules.

Remark. As noted in [Lur2, Remark 7.1.1.16], this statement holds for a discrete ring spectra $\mathcal{R}$.

3.5.3. *Derived functors.* We continue to use the symbols in the previous part. We now discuss functors on $\mathsf{D}_\infty(\mathrm{Mod}_\Lambda(\mathsf{T}))$.

Let T, T' be ∞-topoi, let $\mathcal{A}, \mathcal{A}'$ be sheaves of commutative rings on T, T' respectively, and $(f, f^\sharp) : (\mathsf{T}, \mathcal{A}) \to (\mathsf{T}', \mathcal{A}')$ be a functor of ringed ∞-topoi. Then we have seen in §3.5.1 that there are the tensor functor $\otimes_\Lambda$ and the internal Hom functor $\mathscr{H}om_\Lambda$ on each of the ∞-categories $\mathsf{Mod}_{\mathcal{A}}(\mathsf{T})$ and $\mathsf{Mod}_{\mathcal{A}'}(\mathsf{T}')$. We also have the adjunction $f^* : \mathsf{Mod}_{\mathcal{A}}(\mathsf{T}) \rightleftarrows \mathsf{Mod}_{\mathcal{A}'}(\mathsf{T}') : f_*$ of the inverse image functor f^* and the direct image functor f_*. They are the ordinary functors on the category of sheaves of Λ-modules.

On the other hand, by §3.4.3, we have functors $\otimes_{\mathcal{A}}$, $\mathscr{H}om_{\mathcal{A}}$, f_* and f^* on the ∞-categories $\mathsf{Mod}^{\mathrm{stab}}_{\Lambda}(\mathsf{T})$ and $\mathsf{Mod}^{\mathrm{stab}}_{\Lambda}(\mathsf{T}')$ of stable $\mathcal{A}$-modules. With the view of the fully faithful t-exact functor

$$\mathsf{Mod}_{\mathcal{A}}(\mathsf{T}) \simeq \mathsf{D}_{\infty}(\mathrm{Mod}_{\mathcal{A}}(\mathsf{T}))^{\heartsuit} \longrightarrow \mathsf{D}_{\infty}(\mathrm{Mod}_{\mathcal{A}}(\mathsf{T})) \simeq \mathsf{Mod}^{\mathrm{stab}}_{\mathcal{A}}(\mathsf{T})$$

(Proposition 3.5.3), the functors on stable $\mathcal{A}$-modules are the extensions of those on $\mathcal{A}$-modules, and correspond to the derived functors in the ordinary categorical setting.

Hereafter we change the symbols of the functors on the stable modules by the standard derived functors, as collected in the following proposition.

Proposition 3.5.4. Let T, T' be ∞-topoi, let $\mathcal{A}, \mathcal{A}'$ be sheaves of commutative rings on T, T' respectively, and $(f, f^{\sharp}) : (\mathsf{T}, \mathcal{A}) \to (\mathsf{T}', \mathcal{A}')$ be a functor of ringed ∞-topoi.

(1) The tensor product functor $\otimes_{\mathcal{A}} : \mathrm{Mod}_{\mathcal{A}}(\mathsf{T}) \times \mathsf{Mod}_{\mathcal{A}}(\mathsf{T}) \to \mathrm{Mod}_{\mathcal{A}}(\mathsf{T})$ is a right exact functor in each variable, and has a left t-exact extension

$$\otimes^{\mathrm{L}}_{\mathcal{A}} : \mathsf{D}_{\infty}(\mathrm{Mod}_{\mathcal{A}}(\mathsf{T})) \times \mathsf{D}_{\infty}(\mathrm{Mod}_{\mathcal{A}}(\mathsf{T})) \longrightarrow \mathsf{D}_{\infty}(\mathrm{Mod}_{\mathcal{A}}(\mathsf{T})).$$

(2) Let $\mathcal{M} \in \mathrm{Mod}_{\mathcal{A}}(\mathsf{T})$. Then the internal Hom functor $\mathscr{H}om_{\mathcal{A}}(\mathcal{M}, -) : \mathrm{Mod}_{\mathcal{A}}(\mathsf{T}) \to \mathrm{Mod}_{\mathcal{A}}(\mathsf{T})$ is left exact, and has a right t-exact extension $\mathscr{H}om_{\mathcal{A}}(\mathcal{M}, -) : \mathsf{D}_{\infty}(\mathrm{Mod}_{\mathcal{A}}(\mathsf{T})) \to \mathsf{D}_{\infty}(\mathrm{Mod}_{\mathcal{A}}(\mathsf{T}))$. It yields a bifunctor

$$\mathscr{H}om_{\mathcal{A}}(-, -) : \mathsf{D}_{\infty}(\mathrm{Mod}_{\mathcal{A}}(\mathsf{T}))^{\mathrm{op}} \times \mathsf{D}_{\infty}(\mathrm{Mod}_{\mathcal{A}}(\mathsf{T})) \longrightarrow \mathsf{D}_{\infty}(\mathrm{Mod}_{\mathcal{A}}(\mathsf{T})).$$

We denote it by $\mathscr{H}om_{\mathsf{D}_{\infty}(\mathrm{Mod}_{\mathcal{A}}(\mathsf{T}))}$ if we want to distinguish it from $\mathscr{H}om_{\mathcal{A}}$ on $\mathsf{Mod}_{\mathcal{A}}(\mathsf{T})$.

(3) The direct image functor $f_* : \mathrm{Mod}_{\mathcal{A}}(\mathsf{T}) \to \mathrm{Mod}_{\mathcal{A}}(\mathsf{T}')$ is left exact and has a right t-exact extension

$$f_* : \mathsf{D}_{\infty}(\mathrm{Mod}_{\mathcal{A}}(\mathsf{T})) \longrightarrow \mathsf{D}_{\infty}(\mathrm{Mod}_{\mathcal{A}}(\mathsf{T}')).$$

(4) The inverse image functor $f^* : \mathrm{Mod}_{\mathcal{A}}(\mathsf{T}') \to \mathrm{Mod}_{\mathcal{A}}(\mathsf{T})$ is right exact and has a left t-exact extension

$$f^* : \mathsf{D}_{\infty}(\mathrm{Mod}_{\mathcal{A}}(\mathsf{T}')) \longrightarrow \mathsf{D}_{\infty}(\mathrm{Mod}_{\mathcal{A}}(\mathsf{T})).$$

Remark. Taking the homotopy categories, we recover the derived functors on unbounded derived categories in [KS, §14.4].

We also introduce

Notation 3.5.5. Let T and $\mathcal{A}$ be the same as Proposition 3.5.4. For $\mathcal{M}, \mathcal{N} \in \mathsf{D}_{\infty}(\mathsf{Mod}_{\mathcal{A}}(\mathsf{T}))$ and $n \in \mathbb{Z}$, we set

$$\mathscr{E}xt^n_{\mathcal{A}}(\mathcal{M}, \mathcal{N}) := \pi_0 \, \mathscr{H}om_{\mathsf{D}_{\infty}(\mathsf{Mod}_{\mathcal{A}}(\mathsf{T}))}(\mathcal{M}[-n], \mathcal{N}) \in \mathsf{Mod}_{\mathcal{A}}(\mathsf{T}).$$

3.6. **Open and closed geometric immersions.** In this subsection we introduce the notions of open and closed geometric immersions in an ∞-topos. As for closed geometric immersion, our presentation is just a citation from [Lur1, §7.3].

Hereafter we fix an ∞-topos T.

3.6.1. *Open geometric immersion.* Let $\mathbf{1}_{\mathsf{T}} \in \mathsf{T}$ be a final object (Corollary 1.8.5).

Definition 3.6.1. Denote by $\mathrm{Sub}(\mathbf{1}_{\mathsf{T}})$ the set of equivalence classes of monomorphisms $U \to \mathbf{1}_{\mathsf{T}}$ in T. We regard $\mathrm{Sub}(\mathbf{1}_{\mathsf{T}})$ as a poset under inclusion.

Note that $\mathrm{Sub}(\mathbf{1}_{\mathsf{T}})$ can be identified with the set of equivalence classes of (-1)-truncated objects in T. Also, the poset $\mathrm{Sub}(\mathbf{1}_{\mathsf{T}})$ is independent of the choice of $\mathbf{1}_{\mathsf{T}}$ up to canonical isomorphism by Fact 1.3.9.

Recall that any over-∞-category $\mathsf{T}_{/C}$ of $C \in \mathsf{T}$ is an ∞-topos (Fact 1.8.6). Thus for $U \in \mathrm{Sub}(\mathbf{1}_{\mathsf{T}})$ we can define an ∞-topos $\mathsf{T}_{/U}$ by $\mathsf{T}_{/U} := \mathsf{T}_{/U'}$, where $U' \in \mathsf{T}$ is a representative of U. A different choice of U' will cause an equivalent ∞-topos.

Let us also recall the biadjunction $(j_!, j^*, j_*)$ in Fact 3.1.5 associated to an object $U \in \mathsf{T}$. Here $j_! : \mathsf{T}_{/U} \to \mathsf{T}$ is the canonical functor of the over-∞-category $\mathsf{T}_{/U}$ (Corollary B.3.2), and we have a pair of geometric morphisms of ∞-topoi

$$j_! : \mathsf{T}_{/U} \rightleftarrows \mathsf{T} : j^*, \quad j^* : \mathsf{T} \rightleftarrows \mathsf{T}_{/U} : j_*.$$

Now we introduce the notion of an open geometric immersion.

Definition 3.6.2. A geometric morphism $f : \mathsf{U} \to \mathsf{T}$ corresponding to the adjunction $f^* : \mathsf{T} \rightleftarrows \mathsf{T}' : f_*$ of ∞-topoi is an *open geometric immersion* if there exists $U \in \mathrm{Sub}(\mathbf{1}_{\mathsf{T}})$ such that the composition $\mathsf{U} \xrightarrow{f_*} \mathsf{T} \xrightarrow{j^*} \mathsf{T}_{/U}$ is an equivalence of ∞-categories. We denote an open geometric immersion typically by $f : \mathsf{U} \hookrightarrow \mathsf{T}$.

Following [SGA4] we name the functors appearing in the above argument as

Definition 3.6.3. For an open geometric immersion $j : \mathsf{T}_{/U} \hookrightarrow \mathsf{T}$ with $U \in \mathrm{Sub}(\mathbf{1}_{\mathsf{T}})$, the functor $j_! : \mathsf{T}_{/U} \to \mathsf{T}$ is called the *extension by empty*, and $j^* : \mathsf{T} \to \mathsf{T}_{/U}$ is called the *restriction functor*.

Let us give a few formal properties of an open geometric immersion which is an analogue of [SGA4, IV Proposition 9.2.4].

Lemma 3.6.4. Let $j : \mathsf{U} \hookrightarrow \mathsf{T}$ be an open geometric immersion.
(1) The functors $j_!$ and j_* are fully faithful (Definition 1.3.4).
(2) The counit transformation $j_! j^* \to \mathrm{id}_{\mathsf{T}}$ (Definition B.5.3) is a monomorphism of functors.
(3) We have a monomorphism $j_! \hookrightarrow j_*$ of functors.

Proof. (1) Replace U by $\mathsf{T}_{/U}$ with $U \in \mathrm{Sub}(\mathbf{1}_{\mathsf{T}})$, and denote by $i : U \to \mathbf{1}_{\mathsf{T}}$ the corresponding monomorphism. Then by Corollary B.3.2 on over-∞-categories and by Fact B.9.6 on monomorphisms, we have a commutative diagram

$$\begin{array}{ccc} \mathsf{T}_{/i} & \xrightarrow{p} & \mathsf{T}_{/\mathbf{1}_{\mathsf{T}}} \\ \downarrow & \nearrow & \downarrow{\scriptstyle t} \\ \mathsf{T}_{/U} & \xrightarrow[j_!]{} & \mathsf{T} \end{array}$$

with p fully faithful. By Fact 1.3.8 on final objects, we see that t is a trivial fibration of simplicial sets with respect to Kan model structure (Fact B.1.2). Then we can see that $j_!$ is fully faithful.

As for j_*, the result follows from biadjunction of $(j_!, j^*, j_*)$.

(2) For any $X \in \mathsf{T}$, the object $j_! j^*(X)$ sits in the pullback square

$$\begin{array}{ccc} j_! j^*(X) & \xrightarrow{c} & X \\ \downarrow & & \downarrow \\ U & \xhookrightarrow[i]{} & \mathbf{1}_{\mathsf{T}} \end{array}$$

Then since i is (-1)-truncated we deduce that c is (-1) by [Lur1, Remark 5.5.6.12].

(3) Since j_* is fully faithful if and only if the unit transformation $\mathrm{id}_{\mathsf{U}} \to j^* j_!$ is an equivalence, the statement follows from (1) and (2).

□

Next we give an analogue of [SGA4, XVII, Lemma 5.1.2] in the ordinary topos theory.

Let $f : \mathsf{T}' \to \mathsf{T}$ be a geometric morphism corresponding to the adjunction $f^* : \mathsf{T} \rightleftarrows \mathsf{T}' : f_*$, and $j : \mathsf{U} = \mathsf{T}_{/U} \hookrightarrow \mathsf{T}$ be an open geometric immersion with $U \in \mathrm{Sub}_{\mathbf{1}_{\mathsf{T}}}$. Note that $f^*(\mathbf{1}_{\mathsf{T}}) = \mathbf{1}_{\mathsf{T}'}$ since f^* is left exact so that it commutes with limits. Thus $U' := f^*(U)$ belongs to $\mathrm{Sub}(\mathbf{1}_{\mathsf{T}'})$, Then, setting $\mathsf{U}' := \mathsf{T}_{/U'}$ and denoting by $j' : \mathsf{U}' \hookrightarrow \mathsf{T}'$ the natural open geometric immersion, we have a square

(3.6.1)
$$\begin{array}{ccc} \mathsf{U}' & \xrightarrow{g} & \mathsf{U} \\ {\scriptstyle j'}\downarrow & & \downarrow{\scriptstyle j} \\ \mathsf{T}' & \xrightarrow[f]{} & \mathsf{T} \end{array}$$

in RTop.

Let us give the geometric morphism $g = (g^* : \mathsf{U} \rightleftarrows \mathsf{U}' : g_*)$ explicitly. Using the associated adjunction $j_! : \mathsf{U} \rightleftarrows \mathsf{T} : j^*$ to the geometric morphism j and $j'^* : \mathsf{T}' \rightleftarrows \mathsf{U}' : j'_*$ to j', we set $g^* := (\mathsf{U} \xrightarrow{j_!} \mathsf{T} \xrightarrow{f^*} \mathsf{T}' \xrightarrow{j'^*} \mathsf{U}')$. Since g^* is a composition of left exact functors, it is also left exact. Similarly we set $g_* := (\mathsf{U}' \xrightarrow{j'_*} \mathsf{T}' \xrightarrow{f_*} \mathsf{T} \xrightarrow{j^*} \mathsf{U})$, which is right exact. The commutativity of the square holds by definition, and the adjunction property of (g^*, g_*) is obvious.

We call U the *inverse image* of U' in T.

Lemma 3.6.5. Let $f : \mathsf{T}' \to \mathsf{T}$ be a geometric morphism of ∞-topoi, and $\mathsf{U} \hookrightarrow \mathsf{T}$ be an open geometric immersion. Under the notation in the square (3.6.1) there exists an equivalence $f^* j_! \xrightarrow{\sim} j'_! g^*$ making the following square commutative up to homotopy.

$$\begin{array}{ccc} f^* j_! & \xrightarrow{\sim} & j'_! g^* \\ \downarrow & & \downarrow \\ f^* j_* & \xrightarrow[\alpha]{} & j'_* g^* \end{array}$$

Here α is the *base change morphism* (Definition C.4.1) associated to the square (3.6.1).

Proof. By definition of g^* and Lemma 3.6.4 there is an equivalence $j'_! g^* \xrightarrow{\sim} f^* j_!$. The commutativity can be checked by restricting to U, but it is trivial. □

3.6.2. *Closed geometric immersion.* Next we turn to closed geometric immersions. We begin with

Definition. Let $U \in \mathsf{T}$.

(1) An object $X \in \mathsf{T}$ is *trivial on* U if for any morphism $U' \to U$ in T the mapping space $\mathrm{Map}_{\mathsf{T}}(U', X) \in \mathcal{H}$ is contractible.

(2) We denote by $\mathsf{T}/U \subset \mathsf{T}$ the full sub-∞-category spanned by trivial objects on U.

Let $U \in \mathrm{Sub}(\mathbf{1}_{\mathsf{T}})$. We define a full sub-$\infty$-category $\mathsf{T}/U \subset \mathsf{T}$ by $\mathsf{T}/U := \mathsf{T}/U'$, where $U' \in \mathsf{T}$ is a representative of U. See [Lur1, Lemma 7.3.2.5] for the independence of the choice of U' in this definition. Now we have

Fact ([Lur1, Proposition 7.3.2.3, Lemma 7.3.2.4])**.** For $U \in \mathrm{Sub}(\mathbf{1}_{\mathsf{T}})$, the ∞-category T/U is a localization of T and is an ∞-topos.

Thus there is a functor $\mathsf{T} \to \mathsf{T}/U$ and the following definition makes sense.

Definition 3.6.6 ([Lur1, Definition 7.3.2.6])**.** A geometric morphism $f = (f^* : \mathsf{T} \rightleftarrows \mathsf{T}' : f_*)$ of ∞-topoi is a *closed geometric immersion* if there exists $U \in \mathrm{Sub}(\mathbf{1}_{\mathsf{T}})$ such that $f_* : \mathsf{T}' \to \mathsf{T}$ induces an equivalence $\mathsf{T}' \to \mathsf{T}/U$ of ∞-categories.

Let us remark that in [Lur1] it is just called a closed immersion.

3.7. **Simplicial ∞-topoi and descent theorem.** In this subsection we will give an ∞-analogue of the theory of cohomological descent in [SGA4, $\mathrm{V}^{\mathrm{bis}}$], [O1, §2]. Our discussion will utilize the ∞-category RTop of ∞-topoi and geometric morphisms (Definition 3.1.3). Let us remark that the statements in this subsection are much more general statements than what we need in the later sections.

3.7.1. *K-injective resolutions.* We begin with a restatement of [Sp, §3] in the context of ringed ∞-topoi.

Let $(\mathsf{T}, \mathcal{A})$ be a ringed ∞-topos with $\mathcal{A}$ a sheaf of commutative rings as in §3.5. Let us denote by $\mathrm{Mod}_{\mathcal{A}}(\mathsf{T}) := \mathrm{h}\,\mathsf{Mod}_{\mathcal{A}}(\mathsf{T})$ the homotopy category of the ∞-category of sheaves of $\mathcal{A}$-modules. Recall that $\mathrm{Mod}_{\mathcal{A}}(\mathsf{T})$ is a Grothendieck abelian category. In this part we denote the unbounded derived ∞-category of sheaves of $\mathcal{A}$-modules by

$$\mathsf{D}_{\infty}(\mathsf{T}, \mathcal{A}) := \mathsf{D}_{\infty}(\mathrm{Mod}_{\mathcal{A}}(\mathsf{T})).$$

Recall that $\mathsf{D}_{\infty}(\mathsf{T}, \mathcal{A})$ is stable and equipped with a t-structure whose heart is equivalent to $\mathsf{Mod}_{\mathcal{A}}(\mathsf{T})$. We also denote by

$$\mathsf{D}^-_{\infty}(\mathsf{T}, \mathcal{A}) := \cup_{n \in \mathbb{Z}} \mathsf{D}_{\infty}(\mathsf{T}, \mathcal{A})_{\geq -n} \subset \mathsf{D}_{\infty}(\mathsf{T}, \mathcal{A})$$

the sub-∞-category of right bounded objects. Then by [Sp, 3.6] we have

Lemma 3.7.1. For any $\mathcal{M} \in \mathsf{D}_{\infty}(\mathsf{T}, \mathcal{A})$, there exists a morphism $f : \mathcal{M} \to \mathcal{I}$ in $\mathsf{D}_{\infty}(\mathsf{T}, \mathcal{A})$ satisfying the following conditions.

- $\mathcal{I} = \varinjlim \mathcal{I}_n$ with $\mathcal{I}_n \in \mathsf{D}^-_{\infty}(\mathsf{T}, \mathcal{A})$ and $\pi_j \mathcal{I}_n$ injective for all $n \in \mathbb{N}$, $j \in \mathbb{Z}$.
- f is induced by a compatible collection of equivalences $f_n : \tau_{\geq -n}\mathcal{M} \to \mathcal{I}_n$.
- For each n, the morphism $\mathcal{I}_n \to \mathcal{I}_{n-1}$ is an epimorphism whose kernel K_n belongs to $\mathsf{D}^-_{\infty}(\mathsf{T}, \mathcal{A})$ and $\pi_j K_n$ are injective for any $j \in \mathbb{Z}$.

We now restate [Sp, 3.13 Proposition] and [LO1, 2.1.4 Proposition]. For stating that, let us recall that for an ∞-site (X, τ) we denote by $\mathsf{Sh}(\mathsf{X}, \tau)$ the ∞-category of τ-sheaves on X, which is an ∞-topos (Definition C.1.2, Fact 1.7.4).

Lemma 3.7.2. Let $\mathsf{B} \subset \mathsf{Mod}_{\mathcal{A}}(\mathsf{T})$ a full sub-∞-category. Assume the following conditions for $(\mathsf{T}, \mathcal{A})$ and B.

(S0) T is equivalent to an ∞-topos of the form $\mathsf{Sh}(\mathsf{X}, \tau)$ with some ∞-site (X, τ).

(S1) For each $U \in \mathsf{X}$, there exists a covering $\{U_j \to U\}_{j \in J}$ and an integer n_0 such that we have $\pi_{-n}\, \mathcal{M}|_{U_j} = 0$ for any $\mathcal{M} \in \mathsf{B}$, $n \geq n_0$ and $j \in J$. Here $\mathcal{M}|_{U_j}$ denotes the restriction (Notation 3.1.6).

Let $\mathcal{M} \in \mathsf{D}_\infty(\mathsf{T}, \mathcal{A})$ satisfy $\pi_j \mathcal{M} \in \mathsf{B}$ for all j. Then the morphism $f : \mathcal{M} \to \mathcal{I}$ in Lemma 3.7.1 is an equivalence in $\mathsf{D}_\infty(\mathsf{T}, \mathcal{A})$.

Remark. We can weaken the condition (S0) by the one that T is hypercomplete, but will not pursue this point.

3.7.2. *Simplicial ∞-topoi.* We give some notations for simplicial ∞-topoi. The symbols are borrowed from [LO1, 2.1]. Let us begin with the following citation.

Definition ([Lur1, Definition 6.3.1.6])**.** A map $p : X \to S$ of simplicial sets is a *topos fibration* if the following three conditions are satisfied.

(1) The map p is both a cartesian fibration and a cocartesian fibration (Definition B.5.1).
(2) For every vertex s of S, the corresponding fiber $X_s = X \times_S \{s\}$ is an ∞-topos.
(3) For every edge $e : s \to s_0$ in S, the associated functor $X_s \to X_{s_0}$ is left exact.

Recall the ∞-category RTop of ∞-topoi and geometric morphisms (Definition 3.1.3). By [Lur1, Theorem 6.3.3.1], RTop admits small filtered limits. Let us explain some relevant notions in the proof of loc. cit. for later use.

Let I be a small filtered ∞-category and $q : \mathsf{I}^{\mathrm{op}} \to \mathsf{RTop}$ be any functor of ∞-categories. Then by [Lur1, Proposition 6.3.1.7], there is a topos fibration $p : X \to \mathsf{I}^{\mathrm{op}}$ classified by q (see [Lur1, Definition 3.3.2.2] for the definition of a classifying functor of cartesian fibration).

By [Lur1, Proposition 5.3.1.18], for any filtered ∞-category I, we have a filtered poset A and a cofinal map $\mathsf{N}(A) \to \mathsf{I}$ of simplicial sets. Here A is regarded as a category in which the set of morphisms is given by

$$\mathrm{Hom}_A(a, b) := \begin{cases} \{*\} & a \leq b, \\ \emptyset & a \not\leq b. \end{cases}$$

See also [Lur1, Definition 4.1.1.1] for the definition of a cofinal map of simplicial sets. Thus, as for the functor $q : \mathsf{I}^{\mathrm{op}} \to \mathsf{RTop}$, we can replace I by $\mathsf{N}(A)$.

Now let us given a functor $q : \mathsf{N}(A)^{\mathrm{op}} \to \mathsf{RTop}$ and $p : X \to \mathsf{N}(A)^{\mathrm{op}}$ be a topos fibration classified by q. Then by [Lur1, Proposition 6.3.3.3] the simplicial subset $\mathcal{X} \subset \mathrm{Map}(\mathsf{N}(A)^{\mathrm{op}}, X)$ of cartesian sections of $p : X \to \mathsf{N}(A)^{\mathrm{op}}$ is an ∞-topos. Moreover by [Lur1, Proposition 6.3.3.5] we find that for each $a \in A$ the evaluation $\mathcal{X} \to X_a$ gives a geometric morphism of ∞-topoi. Thus $\mathcal{X}$ gives a filtered limit of $q : \mathsf{N}(A)^{\mathrm{op}} \to \mathsf{RTop}$.

Definition. Let I be a filtered ∞-category. An I*-simplicial ∞-topos* is a topos fibration $p : \mathsf{T}_\bullet \to \mathsf{I}^{\mathrm{op}}$ which is classified by a functor $q : \mathsf{I}^{\mathrm{op}} \to \mathsf{RTop}$ of ∞-categories. We will often denote $\mathsf{T}_\bullet$ to indicate an I-simplicial ∞-topos.

Let $\mathsf{T}_\bullet$ be an I-simplicial ∞-topos. For each $\delta : i \to j$ in I, we denote the corresponding geometric morphism of ∞-topoi by the same symbol $\delta : \mathsf{T}_j \to \mathsf{T}_i$, and denote the associated adjunction by $\delta^{-1} : \mathsf{T}_i \rightleftarrows \mathsf{T}_j : \delta_*$. Note that our convention is the opposite of [LO1, 2.1].

By the above argument $\mathsf{T}_\bullet$ is equipped with a geometric morphism $e_i : \mathsf{T}_i \to \mathsf{T}_\bullet$ of ∞-topoi for each $i \in \mathsf{I}$ corresponding to an adjunction $e_i^{-1} : \mathsf{T}_\bullet \rightleftarrows \mathsf{T}_i : e_{i,*}$.

Let us explain examples of I-simplicial ∞-topoi which will be used in the later text. Recall first that $\boldsymbol{\Delta}$ denotes the ∞-category of combinatorial simplices (Definition 1.1.1), and that a contravariant functor of ordinary categories from $\boldsymbol{\Delta}$ is called a simplicial object (§1.1).

Definition 3.7.3. (1) We denote by $\boldsymbol{\Delta}^{\mathrm{str}} \subset \boldsymbol{\Delta}$ the subcategory with the same objects but morphisms only the injective maps. A contravariant functor from $\boldsymbol{\Delta}^{\mathrm{str}}$ is called a *strictly simplicial object*.
(2) A *simplicial ∞-topos* is defined to be a $\boldsymbol{\Delta}$-simplicial ∞-topos. A *strictly simplicial ∞-topos* is defined to be a $\boldsymbol{\Delta}^{\mathrm{str}}$-simplicial ∞-topos.
(3) For a simplicial ∞-topos $\mathsf{T}_\bullet$, we denote by $\mathsf{T}_\bullet^{\mathrm{str}}$ the strictly simplicial ∞-topos obtained by restriction to $\boldsymbol{\Delta}^{\mathrm{str}} \subset \boldsymbol{\Delta}$.

Remark. In the papers [O1, LO1, LO2], the symbol Δ^+ is used for our $\boldsymbol{\Delta}^{\mathrm{str}}$. In [Lur1, Notation 6.5.3.6], the symbol $\boldsymbol{\Delta}_s$ is used and a contravariant functor from $\boldsymbol{\Delta}_s$ is called a semisimplicial object. We will not use these notations.

Another example of an I-simplicial topos is

Definition 3.7.4. Let $\mathbb{N} = \{0, 1, 2 \dots\}$ be the set of non-negative integers, regarded as a filtered poset with the standard order. For an ∞-topos T, we denote by $\mathsf{T}^{\mathbb{N}}$ the $\mathsf{N}(\mathbb{N})$-simplicial ∞-topos associated to the constant functor $q : \mathsf{N}(\mathbb{N}) \to \mathsf{RTop}$, $q(n) := \mathsf{T}$. We call it the *∞-topos of projective systems in* T.

3.7.3. *Simplicial ringed ∞-topoi.* We will introduce notations for simplicial ringed ∞-topoi. For that, let us first consider a sheaf on a simplicial ∞-topoi. Let I be a filtered ∞-category and $\mathsf{T}_\bullet$ be an I-simplicial ∞-topos. Then a sheaf $\mathcal{F}_\bullet$ with values in an ∞-category C on $\mathsf{T}_\bullet$ consists of sheaves $\mathcal{F}_i \in \mathsf{Shv}_{\mathsf{C}}(\mathsf{T}_\bullet)$ for $i \in \mathsf{I}$ and morphisms $\delta^{-1}\mathcal{F}_j \to \mathcal{F}_i$ in $\mathsf{Shv}_{\mathsf{C}}(\mathsf{T}_i)$ for $\delta : i \to j$ in I.

As in the ordinary topos theory, we introduce

Definition 3.7.5. A functor $f : (\mathsf{T}, \mathcal{A}) \to (\mathsf{T}', \mathcal{A}')$ of ringed ∞-topoi is *flat* if the inverse image functor $f^* : \mathsf{Mod}_{\mathcal{A}'}(\mathsf{T}') \to \mathsf{Mod}_{\mathcal{A}}(\mathsf{T})$ is left and right exact.

Now we have

Definition. Let I be a filtered ∞-category. An *I-simplicial ringed ∞-topos* is a pair $(\mathsf{T}_\bullet, \mathcal{A}_\bullet)$ of an I-simplicial ∞-topos $\mathsf{T}_\bullet$ and a sheaf $\mathcal{A}_\bullet \in \mathsf{Shv}_{\mathsf{CAlg(C)}}(\mathsf{T}_\bullet)$ such that the functor $\delta : (\mathsf{T}_j, \mathcal{A}_j) \to (\mathsf{T}_i, \mathcal{A}_i)$ of ringed ∞-topoi is flat for each $\delta : i \to j$ in I.

As in the non-ringed case, we have a geometric morphism $e_i : (\mathsf{T}_i, \mathcal{A}_i) \to (\mathsf{T}_\bullet, \mathcal{A}_\bullet)$ of ∞-topoi for each $i \in \mathsf{I}$, which corresponds to an adjunction $e_i^{-1} : (\mathsf{T}_\bullet, \mathcal{A}_\bullet) \rightleftarrows (\mathsf{T}_i, \mathcal{A}_i) : e_{i,*}$.

Under these preliminaries let us explain the results in [LO1, 2.1]. Let I be a filtered ∞-category and $(\mathsf{T}_\bullet, \mathcal{A}_\bullet)$ be an I-simplicial ringed ∞-topos. We assume that $\mathcal{A}_\bullet$ is a sheaf of commutative rings. Then the homotopy category $\mathrm{Mod}_{\mathcal{A}_\bullet}(\mathsf{T}_\bullet) := \mathrm{h}\,\mathsf{Mod}_{\mathcal{A}_\bullet}(\mathsf{T}_\bullet)$ is a Grothendieck abelian category. Thus the following definition makes sense.

Notation 3.7.6. We denote the associated derived ∞-category by

$$\mathsf{D}_\infty(\mathsf{T}_\bullet, \mathcal{A}_\bullet) := \mathsf{D}_\infty(\mathrm{Mod}_{\mathcal{A}_\bullet}(\mathsf{T}_\bullet)).$$

Let also $\mathsf{B}_\bullet$ be a full sub-∞-category of $\mathsf{Mod}_{\mathcal{A}_\bullet}(\mathsf{T}_\bullet)$, and for each $i \in \mathsf{I}$ let B_i be the essential image of $\mathsf{B}_\bullet$ under the geometric morphism $e_i : (\mathsf{T}_i, \mathcal{A}_i) \to (\mathsf{T}_\bullet, \mathcal{A}_\bullet)$.

Now Lemma 3.7.2 and the argument [LO1, 2.1.9 Proposition] yield

Lemma 3.7.7. Assume the following condition on $(\mathsf{T}_\bullet, \mathcal{A}_\bullet)$ and $\mathsf{B}_\bullet$.

(S2) For each $i \in \mathsf{I}$, the ringed ∞-topos $(\mathsf{T}_i, \mathcal{A}_i)$ and the ∞-category B_i satisfy the assumptions (S0) and (S1) in Lemma 3.7.2

Then for each $\mathcal{M} \in \mathsf{D}_\infty(\mathsf{T}_\bullet, \mathcal{A}_\bullet)$ there exists a morphism $\mathcal{M} \to \mathcal{I}$ in $\mathsf{D}_\infty(\mathsf{T}_\bullet, \mathcal{A}_\bullet)$ satisfying the conditions in Lemma 3.7.1. If moreover $\pi_j\mathcal{M} \in \mathsf{B}_\bullet$ for any $j \in \mathbb{Z}$, then f is an equivalence.

3.7.4. *Descent lemma.* In this part we review the result in [LO1, 2.2].

Let $(\mathsf{S}, \mathcal{B})$ be a ringed ∞-topos with $\mathcal{B}$ a sheaf of commutative rings, i.e., $\mathcal{B} \in \mathsf{Shv}_{\mathsf{Com(Ab)}}(\mathsf{S})$. As in §3.7.1, we denote the derived ∞-category of sheaves of stable $\mathcal{B}$-modules on S by $\mathsf{D}_\infty(\mathsf{S}, \mathcal{B}) := \mathsf{D}_\infty(\mathrm{Mod}_{\mathcal{B}}(\mathsf{S}))$ with $\mathrm{Mod}_{\mathcal{B}}(\mathsf{S}) := \mathrm{h}\,\mathsf{Mod}_{\mathcal{B}}(\mathsf{S})$. We always consider the t-structure on $\mathsf{D}_\infty(\mathsf{S}, \mathcal{B})$ explained in Lemma 3.5.2. In particular we have $\mathsf{D}_\infty(\mathsf{S}, \mathcal{B})^\heartsuit \simeq \mathsf{Mod}_{\mathcal{B}}(\mathsf{S})$.

Let $\mathsf{B}' \subset \mathsf{Mod}_{\mathcal{B}}(\mathsf{S})$ be a sub-∞-category such that the homotopy category $\mathrm{h}\,\mathsf{B}'$ is a Serre subcategory of the abelian category $\mathrm{Mod}_{\mathcal{B}}(\mathsf{S})$. In other words, $\mathrm{h}\,\mathsf{B}'$ is closed under kernels, cokernels and extensions. We denote by

$$\mathsf{D}_{\infty,\mathsf{B}'}(\mathsf{S}, \mathcal{B}) \subset \mathsf{D}_\infty(\mathsf{S}, \mathcal{B})$$

the full sub-∞-category spanned by those $\mathcal{M}$ such that all the homotopy groups $\pi_j\mathcal{M}$ belongs to B'.

Next let $(\mathsf{T}_\bullet, \mathcal{A}_\bullet)$ be a simplicial and strictly simplicial ringed ∞-topoi respectively (Definition 3.7.3) with $\mathcal{A}_i$ a sheaf of commutative rings for each $i \in \mathsf{I}$ ($\mathsf{I} = \mathsf{N}(\boldsymbol{\Delta})$ or $\mathsf{I} = \mathsf{N}(\boldsymbol{\Delta}^{\mathrm{str}})$). Let $\epsilon : (\mathsf{T}_\bullet, \mathcal{A}_\bullet) \to (\mathsf{S}, \mathcal{B})$ be a functor of ringed ∞-topoi such that $\epsilon_i : (\mathsf{T}_i, \mathcal{A}_i) \to (\mathsf{S}, \mathcal{B})$ is flat (Definition 3.7.5) for each $i \in \mathsf{I}$.

Let $\mathsf{B}'_\bullet$ be the image of B' under the functor $\epsilon^* : \mathsf{D}_\infty(\mathsf{S}, \mathcal{B})^\heartsuit \to \mathsf{D}_\infty(\mathsf{T}_\bullet, \mathcal{A}_\bullet)^\heartsuit$.

Now the argument in [LO1, 2.2.2 Lemma, 2.2.3 Theorem] gives

Proposition 3.7.8. Assume the following conditions.

(S3) $(\mathsf{T}_\bullet, \mathcal{A}_\bullet)$ and $\mathsf{B}'_\bullet$ satisfy the condition (S2) in Lemma 3.7.7.

(S4) The (restricted) morphism $\epsilon^* : \mathsf{B}' \to \mathsf{B}'_\bullet$ is an equivalence.

Then the homotopy category $\mathrm{h}\,\mathsf{B}'_\bullet$ is a Serre subcategory of $\mathrm{h}\,\mathsf{D}_\infty(\mathsf{T}_\bullet, \mathcal{A}_\bullet)^\heartsuit$, and ϵ^* induces an equivalence

$$\epsilon^* : \mathsf{D}_{\infty,\mathcal{B}}(\mathsf{S}) \longrightarrow \mathsf{D}_{\infty,\mathcal{B}_\bullet}(\mathsf{T}_\bullet)$$

of stable ∞-categories. The quasi-inverse is induced by the right adjoint ε_*.

3.7.5. *Gluing lemma.* In this part we recall the result in [LO1, 2.3].

Let $\mathbf{\Delta}_+^{\mathrm{str}}$ be the category of possibly empty finite ordered sets with injective order preserving maps. Thus the objects of $\mathbf{\Delta}_+^{\mathrm{str}}$ are $[n] = \{0, 1, \dots, n\}$ ($n \in \mathbb{N}$) and $\emptyset$. Hereafter we denote $[-1] := \emptyset$. We can regard $\mathbf{\Delta}^{\mathrm{str}}$ (Definition 3.7.3) as a full subcategory of $\mathbf{\Delta}_+^{\mathrm{str}}$.

Let T be an ∞-topos and $U_\bullet^{\mathrm{str}} \to \emptyset_{\mathsf{T}}$ be a strictly simplicial hypercovering of the initial object $\emptyset_{\mathsf{T}}$ of T (Corollary 1.8.5). We denote the localized ∞-topos $\mathsf{T}_{/U_n}$ on U_n (Fact 1.8.6) by the same symbol U_n for $n \in \mathbb{N}$. We also denote $U_{-1} := \emptyset_{\mathsf{T}}$. Thus we have a strictly simplicial ∞-topos $U_\bullet^{\mathrm{str}}$ with an augmentation $\pi : U_\bullet^{\mathrm{str}} \to \mathsf{T}$.

Let $\mathcal{A}$ be a sheaf of commutative rings on T. We denote the induced sheaf on $U_\bullet^{\mathrm{str}}$ by the same symbol $\mathcal{A}$. Then π induces a functor $(U_\bullet^{\mathrm{str}}, \mathcal{A}) \to (\mathsf{T}, \mathcal{A})$ of ringed ∞-topoi, which will be denoted by the same symbol π.

We now consider the cartesian and cocartesian fibration over $\mathbf{\Delta}_+^{\mathrm{str}}$ whose fiber is given by $\mathsf{Mod}_{\mathcal{A}}(U_n)$. Let $\mathsf{C}_\bullet$ be a full sub-∞-category of this fibration such that $\mathrm{h}\,\mathsf{C}_n$ is a Serre subcategory of the abelian category $\mathrm{h}\,\mathsf{Mod}_{\mathcal{A}}(U_n)$ for each n.

We impose on T, $U_\bullet^{\mathrm{str}}$ and $\mathcal{A}$ the following conditions.

(G1) For any $[n] \in \mathbf{\Delta}_+^{\mathrm{str}}$, the ∞-topos U_n is equivalent to $\mathsf{Sh}(\mathsf{X}_n, \tau)$ with some ∞-site (X_n, τ) where for each $V \in \mathsf{X}_n$ there exists $n_0 \in \mathbb{Z}$ and a covering sieve $\{V_j \to V\}_{j \in J}$ such that for any $\mathcal{M} \in \mathsf{C}_n$ we have $H^q(V_j, \mathcal{M}) = 0$ for any $q \geq n_0$.

(G2) The functor $\mathsf{C}_{-1} \to \{\text{cartesian sections of } \mathsf{C}_\bullet^{\mathrm{str}} \to \mathbf{\Delta}^{\mathrm{str}}\}$ is an equivalence, where $\mathsf{C}_\bullet^{\mathrm{str}} := \mathsf{C}_\bullet|_{\mathbf{\Delta}^{\mathrm{str}}}$ (Definition 3.7.3).

(G3) $\mathsf{D}_\infty(\mathsf{T}, \mathcal{A})$ is compactly generated.

Fact 3.7.9 ([LO1, 2.3.3 Theorem])**.** Let T, $U_\bullet^{\mathrm{str}}$ and $\mathcal{A}$ be as above and assume the conditions (G1)–(G3). Consider the cartesian and cocartesian fibration $\mathsf{D}_\bullet \to \mathbf{\Delta}^{\mathrm{str}}$ whose fiber over $[n] \in \mathbf{\Delta}^{\mathrm{str}}$ is $\mathsf{D}_{\infty, \mathsf{C}_n}(U_n, \mathcal{A})$. Let $[n] \mapsto \mathcal{K}_n \in \mathsf{D}_{\infty, \mathsf{C}_n}(U_n, \mathcal{A})$ be a cartesian section of this fibration $\mathsf{D}_\bullet \to \mathbf{\Delta}^{\mathrm{str}}$ such that $\mathscr{E}xt^i_{\mathcal{A}}(\mathcal{K}_0, \mathcal{K}_0) = 0$ for any $i \in \mathbb{Z}_{<0}$. Then $(\mathcal{K}_n)_{n \in \mathbb{N}}$ determines an object $\mathcal{K} \in \mathsf{D}_{\infty, \mathsf{C}_{-1}}(\mathsf{T}, \mathcal{A})$ such that the natural functor $\mathsf{D}_{\infty, \mathsf{C}_{-1}}(\mathsf{T}, \mathcal{A}) \to \{\text{cartesian sections of the fibration } \mathsf{D}_\bullet \to \Delta^{\mathrm{str}}\}$ recovers $(\mathcal{K}_n)_{n \in \mathbb{N}}$. Moreover $\mathcal{K}$ is determined up to contractible ambiguity.

We will not repeat the proof, but record one useful lemma. It is used in [LO1] to show the uniqueness of the object $\mathcal{K}$ in the above Fact 3.7.9.

Lemma 3.7.10 ([LO1, 2.3.4. Lemma])**.** Let T and $\mathcal{A}$ be as above. Also let $\mathcal{M}$ and $\mathcal{N}$ be objects in $\mathsf{D}_\infty(\mathsf{T}, \mathcal{A})$ satisfying $\mathscr{E}xt^i_{\mathcal{A}}(\mathcal{M}, \mathcal{N}) = 0$ for every $i < 0$. For $U \in \mathsf{T}$, we denote the ∞-topos $\mathsf{T}_{/U}$ by the same symbol U. Then the correspondence

$$U \longmapsto \mathrm{Map}_{\mathsf{D}_\infty(U, \mathcal{A})}(\mathcal{M}_U, \mathcal{L}_U)$$

for each $U \in \mathsf{T}$ determines a unique object of $\mathsf{Mod}_{\mathcal{A}}(\mathsf{T})$.

4. Derived algebraic spaces and étale sheaves

In this section we introduce *derived algebraic spaces*, which are derived analogue of algebraic spaces, and étale sheaves on them. These will be used to define the lisse-étale sheaves on derived stacks in §5.

Let us explain a little bit more the motivation for introducing derived algebraic spaces. Our definition of the lisse-étale ∞-site for a derived stack will be a direct analogue of the lisse-étale site for an algebraic stack (see §A.3 for a brief recollection). In the non-derived case, one can describe every object on the lisse-étale site of an algebraic stack X in terms of an object on the big étale site of the simplicial algebraic space $X_\bullet$ making out of a smooth étale covering $U \to X$, as explained in [O1, O2]. Thus it will be useful for the study of the lisse-étale ∞-site to introduce a derived analogue of algebraic spaces.

In this section we work over a fixed base commutative ring k unless otherwise stated. We suppress the notation of k unless confusion would arise. For example, dSt means dSt_k.

4.1. **Derived algebraic spaces.**

4.1.1. *Geometric derived stacks as quotients.* We begin with the citation from [TVe2, §1.3.4] on a characterization of a geometric derived stack as a quotient of Segal groupoid. Our presentation is a translation of the model-theoretic statements in loc. cit. to the ∞-theoretic language.

Recall the category $\mathbf{\Delta}$ of combinatorial simplices (Definition 1.1.1), and let $X_\bullet : \mathbf{\Delta}^{\mathrm{op}} \to \mathsf{C}$ be a simplicial object in an ∞-category C. For $n \in \mathbb{Z}_{>0}$ and $i = 0, 1, \dots, n-1$, we define a morphism

$$\sigma_i : X_n \longrightarrow X_1$$

in C to be the pullback by the order-preserving map $[1] \longrightarrow [n]$ given by $0 \mapsto i$ and $1 \mapsto i+1$. We denote the face map (§1.1) of $X_\bullet$ by

$$d_j : X_n \longrightarrow X_{n-1}$$

for $n \in \mathbb{N}$ and $j = 0, \dots, n-1$. It is defined to be the pullback by the order-preserving map $[n-1] \to [n]$ with $i \mapsto i$ for $i < j$ and $i \mapsto i+1$ for $i \geq j$.

Definition 4.1.1. A *Segal groupoid object* in an ∞-category C is a simplicial object $X_\bullet : \mathbf{\Delta}^{\mathrm{op}} \to \mathsf{C}$ satisfying the following two conditions.

- For any $n \in \mathbb{Z}_{>0}$, the following morphism is an equivalence in C.

$$\prod_{i=0}^{n-1} \sigma_i : X_n \longrightarrow \underbrace{X_1 \times_{d_0, X_0, d_0} X_1 \times_{d_0, X_0, d_0} \cdots \times_{d_0, X_0, d_0} X_1}_{n\text{-times}}.$$

- The morphism $d_0 \times d_1 : X_2 \to X_1 \times_{d_0, X_0, d_0} X_1$ is an equivalence in C.

Definition. For $n \in \mathbb{Z}_{\geq -1}$, a Segal groupoid object $\mathcal{X}_\bullet$ in dSt_k is *n-smooth* if it satisfies the following two conditions.

- The derived stacks $\mathcal{X}_0$ and $\mathcal{X}_1$ are (small) coproduct of n-geometric derived stacks.
- The morphism $d_0 : \mathcal{X}_1 \to \mathcal{X}_0$ of derived stacks is n-smooth in the sense of Definition 2.2.14.

Fact 4.1.2 ([TVe2, Proposition 1.3.4.2])**.** For a derived stack $\mathcal{X}$ and $n \in \mathbb{N}$, the following two conditions are equivalent.

(i) $\mathcal{X}$ is n-geometric.
(ii) There exists an $(n-1)$-smooth Segal groupoid object $\mathcal{X}_\bullet$ in dSt_k such that $\mathcal{X}_0$ is a coproduct of affine derived schemes and $\mathcal{X} \simeq \varinjlim_{[m] \in \mathbf{\Delta}} \mathcal{X}_m$ in dSt_k.

Notation. If one of the conditions in Fact 4.1.2 is satisfied, then we say $\mathcal{X}$ *is the quotient stack of the $(n-1)$-smooth Segal groupoid* $\mathcal{X}_\bullet$.

Note that the expression $\varinjlim_{[n] \in \mathbf{\Delta}} \mathcal{X}_n$ in (ii) makes sense, for dSt_k is an ∞-topos (Fact 2.2.9) so that it admits small limits and colimits (Corollary 1.8.4).

Remark 4.1.3. Let us explain the indication (i) $\Longrightarrow$ (ii), i.e., a construction of a Segal groupoid object $X_\bullet$ from a given n-geometric derived stack $\mathcal{X}$. Let $\{V_i\}_{i \in I}$ be an n-atlas of $\mathcal{X}$. Then we put $\mathcal{X}_0 := \coprod_{i \in I} V_i$, and let $p : \mathcal{X}_0 \to \mathcal{X}$ be the natural effective epimorphism of derived stacks. For $n \geq 1$, we set

$$\mathcal{X}_n := \underbrace{\mathcal{X}_0 \times_{p, \mathcal{X}, p} \mathcal{X}_0 \times_{p, \mathcal{X}, p} \cdots \times_{p, \mathcal{X}, p} \mathcal{X}_0}_{n\text{-times}}.$$

4.1.2. *Derived algebraic spaces.* Recall Fact 2.1.11 which characterizes schemes and algebraic spaces among higher Artin stacks. Considering its simple analogue, we introduce

Definition 4.1.4. (1) A *derived scheme* X (over k) is derived stack having an n-atlas $\{U_i\}_{i \in I}$ with some $n \in \mathbb{Z}_{\geq -1}$ such that each morphism $U_i \to X$ is a monomorphism in dSt (Definition 2.2.11). We denote by $\mathsf{dSch} \subset \mathsf{dSt}$ the sub-∞-category spanned by derived schemes.

(2) A *derived algebraic space* $\mathcal{U}$ (over k) is a derived stack satisfying the following conditions.
  - There exists an n-atlas $\{V_i\}_{i \in I}$ of $\mathcal{U}$ for some $n \in \mathbb{Z}_{\geq -1}$ such that each morphism $V_i \to \mathcal{U}$ of derived stacks is étale.
  - The diagonal morphism $\mathcal{U} \to \mathcal{U} \times \mathcal{U}$ is a monomorphism in dSt.

We denote by $\mathsf{dAS} \subset \mathsf{dSt}$ the sub-∞-category spanned by derived algebraic spaces.

We have the sequence $\mathsf{dAff} \subset \mathsf{dSch} \subset \mathsf{dAS} \subset \mathsf{dSt}$ of ∞-categories. Moreover we have an analogous result to Fact 2.1.9.

Proposition. A derived algebraic space is 1-geometric. In particular, a derived scheme is 1-geometric.

Proof. Let $\mathcal{U}$ be a derived algebraic space and take an n-atlas $\{V_i\}_{i \in I}$. By Remark 2.2.15 (2), we see that $\mathcal{U}$ is n-geometric. By Fact 4.1.2 and Remark 4.1.3, we may assume that $\mathcal{U}$ is a quotient stack of $(n-1)$-smooth Segal groupoid $U_\bullet$ with $U_0 := \coprod_{i \in I} V_i$ and $U_1 := U_0 \times_{\mathcal{U}} U_0$. Then the morphism $d_0 \times d_1 : U_1 \to U_0 \times U_0$ is a monomorphism, where we denoted by $d_0, d_1 : U_1 \to U_0$ the structure morphisms in $U_\bullet$. In fact, the diagram

$$\begin{array}{ccc} U_1 & \xrightarrow{d_0 \times d_1} & U_0 \times U_0 \\ \downarrow & & \downarrow \\ \mathcal{U} & \xrightarrow[\Delta_{\mathcal{U}}]{} & \mathcal{U} \times \mathcal{U} \end{array}$$

is a cartesian square in the ∞-category dSt (Definition B.4.5). Since the diagonal morphism $\Delta_{\mathcal{U}}$ is assumed to be a monomorphism, the morphism $d_0 \times d_1$ is also a monomorphism as claimed above. We can also check that $\{V_i\}_{i \in I}$ is an n-atlas of U_1 with each $V_i \to U_1$ étale. Thus U_1 is a derived algebraic stack. Now note that U_1 is an $(n-1)$-geometric derived stack since $U_\bullet$ is an $(n-1)$-smooth Segal groupoid. Thus U_1 is an $(n-1)$-geometric derived algebraic space, A similar argument shows that U_0 is also an $(n-1)$-geometric derived algebraic spaces. Moreover, as U_0 being a coproduct of affine derived schemes, it is 0-geometric by Lemma 2.2.16. Then from $U_1 \subset U_0 \times U_0$, we see that U_1 is also 0-geometric. Thus $U_\bullet$ is a 0-smooth Segal groupoid, so that the quotient stack $\mathcal{U}$ of $U_\bullet$ is 1-geometric. □

Thus the following definition makes sense.

Definition. A morphism $f : \mathcal{U} \to \mathcal{V}$ in dAS is *smooth* if it is 1-smooth in the sense of Definition 2.2.14.

Remark 4.1.5. Recall the fully faithful functor $\mathrm{Dex} : \mathsf{St} \to \mathsf{dSt}$ (Definition 2.2.27). The image of an algebraic space under Dex is obviously a derived algebraic space, and restricting Dex to $\mathsf{AS} \subset \mathsf{St}$ we have a fully faithful functor

$$\mathrm{Dex} : \mathsf{AS} \longrightarrow \mathsf{dAS}.$$

Similarly, the image of a scheme under Dex is a derived scheme. Thus we have the following diagram (compare with Remark 2.1.12).

$$\begin{array}{ccccccc} \mathsf{Aff} & \hookrightarrow & \mathsf{Sch} & \hookrightarrow & \mathsf{AS} & \hookrightarrow & \mathsf{St}_{\mathrm{geom}} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \mathrm{Dex} \\ \mathsf{dAff} & \hookrightarrow & \mathsf{dSch} & \hookrightarrow & \mathsf{dAS} & \hookrightarrow & \mathsf{dSt}_{\mathrm{geom}} \end{array}$$

4.2. **Étale ∞-site on a derived algebraic space.**

4.2.1. *∞-sites and ∞-topoi.* Here we give some complementary explanation on ∞-topoi arising from ∞-sites. We will give a construction of geometric morphisms of ∞-sites from a *continuous* functor of ∞-sites.

Definition 4.2.1. A *continuous functor* $f : (\mathsf{C}', \tau') \to (\mathsf{C}, \tau)$ of ∞-sites is a functor $\mathsf{C}' \to \mathsf{C}$ of ∞-categories satisfying the following two conditions.

- For every $X' \in \mathsf{C}'$ and $\{X'_i \to X'\}_{i \in I} \in \mathrm{Cov}_{\tau'}(X')$, the family $\{f(X'_i) \to f(X')\}_{i \in I}$ is in $\mathrm{Cov}_\tau(f(X'))$.
- f commutes with finite limits (if they exist).

Restating the argument in [O2, §2.2], we have

Proposition 4.2.2 ([O2, Proposition 2.2.26, 2.2.31])**.** Let $f : (\mathsf{C}', \tau') \to (\mathsf{C}, \tau)$ be a functor of ∞-sites.

(1) The composition with f induces a functor $f_* : \mathsf{Sh}(\mathsf{C}, \tau) \to \mathsf{Sh}(\mathsf{C}', \tau')$ of ∞-topoi.
(2) If f is continuous, then f_* has a left adjoint $f^* : \mathsf{Sh}(\mathsf{C}', \tau') \to \mathsf{Sh}(\mathsf{C}, \tau)$.
(3) If C and C' admit finite limits and if f is continuous and commutes with finite limits, then f^* is left exact and the adjunction $f^* : \mathsf{Sh}(\mathsf{C}', \tau') \rightleftarrows \mathsf{Sh}(\mathsf{C}, \tau) : f_*$ determines a geometric morphism $\mathsf{Sh}(\mathsf{C}, \tau) \to \mathsf{Sh}(\mathsf{C}', \tau')$.

We followed Remark 3.1.2 on the notation of geometric morphisms.

Proof. These statements are essentially shown in [O2, §2.2], but let us explain an outline of the proof. The proof of (1) is standard, so we omit it.

Let us explain the construction of f^* in (2). Given a sheaf $\mathcal{G} \in \mathsf{Sh}(\mathsf{C}', \tau')$, we want to construct $f^*\mathcal{G} \in \mathsf{Sh}(\mathsf{C}, \tau)$. For an object $U \in \mathsf{C}$, we denote by I_U the full sub-∞-category of the under-∞-category $\mathsf{C}_{U/}$ spanned by the essential image of the functor f. An object of I_U can be regarded as a pair (V, ν) where $V \in \mathsf{C}'$ and $\nu : U \to f(V)$ is a morphism in C. Now we define $\mathcal{F} \in \mathsf{PSh}(\mathsf{C})$ by $\mathcal{F}(U) := \varinjlim_{(V,\nu) \in \mathsf{I}_U} \mathcal{G}(V)$. It is immediate that $\mathcal{F}$ is well-defined, and we define the sheaf $f^*\mathcal{G}$ to be the sheafification of $\mathcal{F}$.

As for the item (3), since the localization functor in a topological localization is left exact [Lur1, Corollary 6.1.2.6], the sheafification functor is left exact by Fact 1.7.4. Then the construction implies that f^* is left exact. By the same argument in the proof of [O2, Proposition 2.2.31], we have a counit transformation $f_* f_* \to \mathrm{id}$ (Definition B.5.3). Considering the evaluation at the object $(V, \mathrm{id}_{f(V)})$ of $\mathsf{I}_f(V)$ for any $V \in \mathsf{Sh}(\mathsf{C}', \tau')$, we obtain the inverse of the counit transformation. Thus we have the desired adjunction $f^* : \mathsf{Sh}(\mathsf{C}', \tau') \rightleftarrows \mathsf{Sh}(\mathsf{C}, \tau) : f_*$. □

4.2.2. *Étale ∞-site.* Here we introduce an ∞-theoretical analogue of étale topoi of schemes. We continue to work over a commutative ring k.

Definition 4.2.3. Let $\mathcal{U}$ be a derived algebraic space.

(1) The (*small*) *étale ∞-site* of $\mathcal{U}$ is the ∞-site $\mathrm{Et}(\mathcal{U}) := (\mathsf{dAS}^{\mathrm{et}}_{\mathcal{U}}, \mathrm{et})$ consisting of
  - The full sub-∞-category $\mathsf{dAS}^{\mathrm{et}}_{\mathcal{U}}$ of the over-∞-category $\mathsf{dAS}_{/\mathcal{U}}$ spanned by étale morphisms $\mathcal{T} \to \mathcal{U}$ of derived stacks (Definition 2.2.21).
  - The Grothendieck topology et, called the (*small*) *étale topology*, on $\mathsf{dAS}^{\mathrm{et}}_{\mathcal{U}}$ whose set $\mathrm{Cov}_{\mathrm{et}}(\mathcal{T})$ of covering sieves of $\mathcal{T} \in \mathsf{dAS}^{\mathrm{et}}_{\mathcal{U}}$ consists of families $\{\mathcal{T}_i \to \mathcal{T}\}_{i \in I}$ with $\coprod_{i \in I} \mathcal{T}_i \to \mathcal{T}$ an epimorphism of derived stacks (Definition 2.2.11).

(2) We denote the associated ∞-topos (Fact 1.7.4) by $\mathcal{U}_{\mathrm{et}} := \mathsf{Sh}(\mathrm{Et}(\mathcal{U}))$ and call it the (*small*) *étale ∞-topos* on $\mathcal{U}$.

The following statement is an analogue of the one in the étale topology of a scheme (see [O2, Example 2.2.10] for example). The proof is quite similar, and we omit it.

Lemma 4.2.4. For a derived algebraic space $\mathcal{U}$, let $\mathrm{Et}^{\mathrm{aff}}(\mathcal{U}) := (\mathsf{dAff}^{\mathrm{et}}_{\mathcal{U}}, \mathrm{et}^{\mathrm{aff}})$ be the ∞-site consisting of
- The full sub-∞-category $\mathsf{dAff}^{\mathrm{et}}_{\mathcal{U}}$ of the over-∞-category $\mathsf{dAff}_{/\mathcal{U}}$ of *affine derived schemes* over $\mathcal{U}$ spanned by étale morphisms of derived stacks.
- The Grothendieck topology $\mathrm{et}^{\mathrm{aff}}$ where a covering sieve is set to be a covering sieve in et (Definition 4.2.3 (1)).

Then the associated ∞-topos $\mathcal{U}_{\mathrm{et}^{\mathrm{aff}}}$ is equivalent to $\mathcal{U}_{\mathrm{et}}$.

The étale ∞-topos on a derived algebraic space is functorial in the following sense. Let $f : \mathcal{U} \to \mathcal{V}$ be a morphism of derived algebraic spaces. Taking fiber products, we have the following continuous functor of ∞-sites (Definition 4.2.1).

$$f^{-1} : \mathrm{Et}(\mathcal{V}) \longrightarrow \mathrm{Et}(\mathcal{U}), \quad \mathcal{V}' \longmapsto \mathcal{V}' \times_{\mathcal{V}} \mathcal{U}.$$

Composition with f^{-1} gives the following functor of ∞-topoi.

$$f^{\mathrm{et}}_* : \mathcal{U}_{\mathrm{et}} \longrightarrow \mathcal{V}_{\mathrm{et}}, \quad (f^{\mathrm{et}}_* \mathcal{F})(V) := \mathcal{F}(f^{-1}(V)).$$

Applying Proposition 4.2.2 to the present situation, we have

Lemma 4.2.5. The functor $f^{\mathrm{et}}_* : \mathcal{U}_{\mathrm{et}} \to \mathcal{V}_{\mathrm{et}}$ has a left exact left adjoint f^{-1}_{et}. Thus we have a geometric morphism $f_{\mathrm{et}} : \mathcal{U}_{\mathrm{et}} \to \mathcal{V}_{\mathrm{et}}$ of ∞-topoi (Definition 3.1.1) corresponding to the adjunction

$$f^{-1}_{\mathrm{et}} : \mathcal{V}_{\mathrm{et}} \rightleftarrows \mathcal{U}_{\mathrm{et}} : f^{\mathrm{et}}_*.$$

Let us now introduce some basic notions on derived algebraic spaces. These are simple analogue of the corresponding notions in the scheme theory. For our definitions, morphisms of schemes and algebraic spaces will be replaced by *geometric* morphisms of the étale ∞-topoi on derived algebraic spaces. In this part, $\mathcal{U}$ and $\mathcal{V}$ denote derived algebraic spaces over k, and $f : \mathcal{U} \to \mathcal{V}$ denotes a morphism between them.

We begin with the definition of open immersions by applying the discussion in 3.6 for the summary. Our definition is an analogue of [Lur7, Definition 9.5].

Definition 4.2.6. $f : \mathcal{U} \to \mathcal{V}$ is an *open immersion* if the geometric morphism $f_{\mathrm{et}} : \mathcal{U}_{\mathrm{et}} \to \mathcal{V}_{\mathrm{et}}$ of the étale ∞-topoi is an open geometric immersion (Definition 3.6.2).

Let us spell out this definition differently. For an ∞-topos T, we denote by $\mathrm{Sub}(\mathbf{1}_{\mathsf{T}})$ the set of equivalence classes of (-1)-truncated objects in T (Definition 3.6.1). Applying this notation to the étale ∞-topos $\mathsf{T} = \mathcal{U}_{\mathrm{et}}$, each $U \in \mathrm{Sub}(\mathbf{1}_{\mathcal{U}_{\mathrm{et}}})$ is represented by an étale morphism $j_U : U' \to \mathcal{U}$ of derived algebraic spaces where U' is an affine derived scheme.

Then we have that $f : \mathcal{U} \to \mathcal{V}$ is an open immersion if there is a $V \in \mathrm{Sub}(\mathbf{1}_{\mathcal{V}_{\mathrm{et}}})$ represented by an étale morphism $j_V : V' \to \mathcal{V}$ such that f is equivalent to the composition $\mathcal{U} \xrightarrow{f'} V \xrightarrow{j_V} \mathcal{V}$ with f' an equivalence of derived algebraic spaces.

Next we introduce closed immersions by adapting the general notion in [Lur1, §7.2.3]. See 3.6 for the summary. Let us also refer [Lur9, §4] for the relevant discussion in the spectral algebraic geometry.

Definition. $f : \mathcal{U} \to \mathcal{V}$ is a *closed immersion* if the geometric morphism $f_{\mathrm{et}} : \mathcal{U}_{\mathrm{et}} \to \mathcal{V}_{\mathrm{et}}$ of the étale ∞-topoi is a closed geometric immersion (Definition 3.6.6).

Following [Lur9, Definition 1.4.11], we introduce

Definition 4.2.7. (1) $f : \mathcal{U} \to \mathcal{V}$ is *strictly separated* if the diagonal morphism $\mathcal{U} \to \mathcal{U} \times_{f,\mathcal{V},f} \mathcal{U}$ is a closed immersion.
(2) $\mathcal{U}$ is *separated* if the structure morphism $\mathcal{U} \to \operatorname{dSpec} k$ is strictly separated.

We now introduce *quasi-compact* derived algebraic spaces. Recall the notion of quasi-compact ∞-topoi (Definition 1.8.12).

Definition 4.2.8. A derived algebraic space $\mathcal{U}$ is *quasi-compact* if the ∞-topoi $\mathcal{U}_{\mathrm{et}}$ is quasi-compact.

Next we introduce *quasi-compact morphisms* of derived algebraic spaces. For that, we need the notion of quasi-compact morphisms between ∞-topoi. Recall Definition 1.8.12 of quasi-compactness for objects in ∞-topoi.

Definition 4.2.9. A geometric morphism $f : \mathsf{T} \to \mathsf{T}'$ of ∞-topoi is called a *quasi-compact morphism* if for any quasi-compact object $U \in \mathsf{T}'$ the object $f^*U \in \mathsf{T}$ is quasi-compact.

Remark. Starting with quasi-compactness, one can introduce by induction the notion of n-coherence of ∞-topoi [Lur7, §3]. Using the n-coherence, Lurie introduced in [Lur8, §1.4] the notion of n-quasi-compactness for spectral schemes, spectral Deligne-Mumford stacks and morphisms between them. Our definition of quasi-compact morphism is an adaptation of this n-quasi-compactness to the case $n = 0$.

Definition 4.2.10. A morphism $f : \mathcal{U} \to \mathcal{V}$ of derived algebraic spaces is *quasi-compact* if the geometric morphism $f_{\mathrm{et}} : \mathcal{U}_{\mathrm{et}} \to \mathcal{V}_{\mathrm{et}}$ of ∞-topoi (Lemma 4.2.5) is a quasi-compact morphism (Definition 4.2.9).

Remark. We have already introduced the notion of quasi-compact morphisms of derived stacks in Definition 2.2.12. By a routine one can show that these two notions are equivalent.

Let us now turn to the notion of *quasi-separated* derived algebraic spaces. It is a direct analogue of the notion of quasi-separated schemes and quasi-separated algebraic spaces. Let us also refer [Lur12, Definition 1.3.1] for a relevant notion for spectral Deligne-Mumford stacks.

Definition 4.2.11. (1) f is *quasi-separated* if the diagonal morphism $\mathcal{U} \to \mathcal{U} \times_{f,\mathcal{V},f} \mathcal{U}$ is quasi-compact (Definition 4.2.10).
(2) $\mathcal{U}$ is *quasi-separated* if the structure morphism $\mathcal{U} \to \operatorname{dSpec} k$ is quasi-separated in the sense of (1).

Finally we introduce proper morphisms. Our definition is an analogue of the strongly proper morphism in spectral algebraic geometry [Lur12, §3].

Recall the truncation functor $\operatorname{Trc} : \mathrm{h}\,\mathsf{dSt} \to \mathrm{h}\,\mathsf{St}$ (Definition 2.2.27). For a derived algebraic space $\mathcal{U}$, the truncation $\operatorname{Trc} \mathcal{U}$ is an algebraic space.

Definition 4.2.12. (1) A morphism $f : \mathcal{U} \to \mathcal{V}$ of derived algebraic spaces is *proper* if the corresponding morphism $\operatorname{Trc} \mathcal{U} \to \operatorname{Trc} \mathcal{V}$ of algebraic spaces is proper (Definition A.1.10).
(2) A derived algebraic space $\mathcal{U}$ is *proper* if the structure morphism $\mathcal{U} \to \operatorname{dSpec} k$ is proper in the sense of (1).

One can check the ordinary properties of proper morphisms, such as stable under composition and base change, hold in dAS.

Remark 4.2.13. The notions on morphisms of derived algebraic spaces given above and those of ordinary algebraic spaces (§A.1) are compatible under the functor Dex in Remark 4.1.5.

property of morphisms	derived algebraic spaces	algebraic spaces
separated	Definition 4.2.7	Definition A.1.5
quasi-separated	Definition 4.2.11	Definition A.1.6
proper	Definition 4.2.12	Definition A.1.10

TABLE 4.1. Morphisms between derived and algebraic spaces

4.3. **Étale sheaves of rings and modules.** In this subsection we collect notations for sheaves of commutative rings and modules on derived algebraic spaces in the étale topology. We work over the base commutative ring k as before.

4.3.1. *Étale structure sheaves.* This part will not be used in the later sections. We record it for completeness of our presentation.

Definition. For a derived algebraic space $\mathcal{U}$, the (*small*) *étale structure sheaf* $\mathcal{O}_{\mathcal{U}}$ *of* $\mathcal{U}$ is an object of $\mathsf{Shv}_{\mathsf{sCom}_k}(\mathcal{U}_{\mathrm{et}})$ determined by

$$\mathcal{O}_{\mathcal{U}}(U) := A \in \mathsf{sCom}_k \quad \text{for} \quad U = \mathrm{dSpec}\, A \in \mathcal{U}_{\mathrm{et}}.$$

Here $U \in \mathcal{U}_{\mathrm{et}}$ means that $U \in \mathsf{dAS}^{\mathrm{et}}_{\mathcal{U}}$ and it is identified with an object of $\mathcal{U}_{\mathrm{et}}$ by the Yoneda embedding $j(U) \in \mathsf{Sh}(\mathsf{dAS}^{\mathrm{et}}_{\mathcal{U}}, \mathrm{et}) = \mathcal{U}_{\mathrm{et}}$.

Thus we obtain a ringed ∞-topos $(\mathcal{U}_{\mathrm{et}}, \mathcal{O}_{\mathcal{U}})$. It is intimately related to *spectral algebraic spaces* in Lurie's spectral algebraic geometry [Lur7, Lur8]. We also have the stable ∞-category $\mathsf{Mod}_{\mathcal{O}_{\mathcal{U}}}(\mathsf{Shv}_{\mathsf{sCom}_k}(\mathsf{T}))$ of stable étale sheaves of $\mathcal{O}$-modules over $\mathcal{U}$.

For a morphism $f : \mathcal{U} \to \mathcal{V}$ of derived algebraic spaces, we can construct a functor $(f_{\mathrm{et}}, f^{\sharp}) : (\mathcal{U}_{\mathrm{et}}, \mathcal{O}_{\mathcal{U}}) \to (\mathcal{V}_{\mathrm{et}}, \mathcal{O}_{\mathcal{V}})$ of ringed ∞-topoi, where $f_{\mathrm{et}} : \mathcal{U}_{\mathrm{et}} \to \mathcal{V}_{\mathrm{et}}$ is the geometric morphism in Lemma 4.2.5, and $f^{\sharp} : \mathcal{O}_{\mathcal{V}} \to f_* \mathcal{O}_{\mathcal{U}}$ is a morphism in $\mathsf{Shv}_{\mathsf{sCom}_k}(\mathcal{V}_{\mathrm{et}})$. We can also construct a geometric morphism corresponding to the adjunction $f^*_{\mathrm{et}} : \mathsf{Mod}_{\mathcal{O}_{\mathcal{V}}}(\mathcal{V}_{\mathrm{et}}) \rightleftarrows \mathsf{Mod}_{\mathcal{O}_{\mathcal{U}}}(\mathcal{U}_{\mathrm{et}}) : f^{\mathrm{et}}_*$ of functors between stable ∞-categories,

4.3.2. *Étale sheaf of commutative rings.* Recall that in §3.5 we introduced sheaves of commutative rings over an ∞-topos T. Let us set $\mathsf{T} = \mathcal{U}_{\mathrm{et}}$, the étale ∞-topos of an a derived algebraic space $\mathcal{U}$. Thus, for a sheaf $\mathcal{A} \in \mathsf{Shv}_{\mathsf{Com}}(\mathcal{U}_{\mathrm{et}})$, we have a ringed ∞-topos $(\mathcal{U}_{\mathrm{et}}, \mathcal{A})$. We call such an $\mathcal{A}$ an *étale sheaf of commutative rings on* $\mathcal{U}$.

Let us apply the notations in §3.5.1 to the present situation. We call

$$\mathsf{Mod}_{\mathcal{A}}(\mathcal{U}_{\mathrm{et}}) := \mathsf{Mod}_{\mathcal{A}}(\mathsf{Shv}_{\mathsf{Ab}}(\mathcal{U}_{\mathrm{et}})),$$

the ∞*-category of étale sheaves of* $\mathcal{A}$*-modules on* $\mathcal{U}$. It is equipped with internal Hom functor $\mathscr{H}om_{\mathcal{A}}$ and the tensor functor $\otimes_{\mathcal{A}}$. We also denote by $\mathrm{Mod}_{\mathcal{A}}(\mathcal{U}_{\mathrm{et}}) := \mathrm{h}\,\mathsf{Mod}_{\mathcal{A}}(\mathcal{U}_{\mathrm{et}})$ its homotopy category, which is a Grothendieck abelian category (Proposition 3.5.1).

Next recall the notations in §3.5.2. We denote by

$$\mathsf{Mod}^{\mathrm{stab}}_{\mathcal{A}}(\mathcal{U}_{\mathrm{et}}) := \mathsf{Mod}^{\mathrm{stab}}_{\mathcal{A}}(\mathsf{Shv}_{\mathsf{Sp}}(\mathcal{U}_{\mathrm{et}}))$$

the ∞-category of sheaves of stable $\mathcal{A}$-modules on $\mathcal{U}_{\mathrm{et}}$. It is stable and equipped with a t-structure. It also has the internal Hom functor $\mathscr{H}om_{\mathcal{A}}$ and the tensor functor $\otimes_{\mathcal{A}}$. By Proposition 3.5.3, we also have a t-exact equivalence

$$\mathsf{Mod}^{\mathrm{stab}}_{\mathcal{A}}(\mathcal{U}_{\mathrm{et}}) \simeq \mathsf{D}_{\infty}(\mathrm{Mod}_{\mathcal{A}}(\mathcal{U}_{\mathrm{et}}))$$

of stable ∞-categories, where the right hand side is the derived ∞-category of the Grothendieck abelian category $\mathrm{Mod}_{\mathcal{A}}(\mathcal{U}_{\mathrm{et}})$. Hereafter we mainly discuss in terms of the derived ∞-category, and use the following notation.

Notation. For a derived algebraic space $\mathcal{U}$ and an étale sheaf $\mathcal{A}$ of commutative rings on $\mathcal{U}$, we set

$$\mathsf{D}^*_{\infty}(\mathcal{U}_{\mathrm{et}}, \mathcal{A}) := \mathsf{D}^*_{\infty}(\mathrm{Mod}_{\mathcal{A}}(\mathcal{U}_{\mathrm{et}})) \quad * \in \{\emptyset, +, -, b\}$$

and call it the (resp. left bounded, resp. right bounded, resp. bounded) *derived* ∞*-category of étale sheaves of* $\mathcal{A}$*-modules on* $\mathcal{U}$. For a commutative Λ, we denote by

$$\mathsf{D}^*_{\infty}(\mathcal{U}_{\mathrm{et}}, \Lambda) := \mathsf{D}^*_{\infty}(\mathrm{Mod}_{\Lambda}(\mathcal{U}_{\mathrm{et}})) \quad * \in \{\emptyset, +, -, b\}$$

for the derived ∞-category of étale sheaves of Λ-modules on $\mathcal{U}$ (with some bound condition), where we denote the constant sheaf by the same symbol Λ.

Now assume that we are given a morphism $f : \mathcal{U} \to \mathcal{V}$ of derived algebraic spaces. Then by Lemma 4.2.5 we have a geometric morphism $f_{\mathrm{et}} : \mathcal{U}_{\mathrm{et}} \to \mathcal{V}_{\mathrm{et}}$, so that the argument in §3.5 gives rise to the direct image functors

$$f^{\mathrm{et}}_* : \mathsf{Mod}_{\mathcal{A}}(\mathcal{U}_{\mathrm{et}}) \longrightarrow \mathsf{Mod}_{\mathcal{A}}(\mathcal{V}_{\mathrm{et}}), \quad \mathsf{D}_{\infty}(\mathcal{U}_{\mathrm{et}}, \mathcal{A}) \longrightarrow \mathsf{D}_{\infty}(\mathcal{V}_{\mathrm{et}}, \mathcal{A})$$

and inverse image functors

$$f^*_{\mathrm{et}} : \mathsf{Mod}_{\mathcal{A}}(\mathcal{V}_{\mathrm{et}}) \longrightarrow \mathsf{Mod}_{\mathcal{A}}(\mathcal{U}_{\mathrm{et}}), \quad \mathsf{D}_{\infty}(\mathcal{V}_{\mathrm{et}}, \mathcal{A}) \longrightarrow \mathsf{D}_{\infty}(\mathcal{U}_{\mathrm{et}}, \mathcal{A}).$$

4.3.3. *Proper base change.* Recall Definition 4.2.12 of proper morphisms of derived algebraic spaces. It enable us to translate to derived settings the proper base change theorem in the ordinary scheme theory. In the following sections we will mainly discuss constant sheaves of commutative rings. So, for a commutative ring A, let us denote by the *constant sheaf* valued in A on $\mathcal{U}_{\mathrm{et}}$ (Definition 1.8.9) by the same symbol A.

Lemma 4.3.1. Let Λ be a torsion ring and

$$\begin{array}{ccc} \mathcal{U}' & \xrightarrow{g'} & \mathcal{U} \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{V}' & \xrightarrow[g]{} & \mathcal{V} \end{array}$$

be a cartesian square in dAS with f proper. Then the base change morphism (Definition C.4.1)

$$g_{\mathrm{et}}^{*}\, f_{*}^{\mathrm{et}} \longrightarrow f'^{\mathrm{et}}_{*}\, g'^{*}_{\mathrm{et}}$$

of functors $\mathsf{Mod}_{\Lambda}(\mathcal{U}'_{\mathrm{et}}) \to \mathsf{Mod}_{\Lambda}(\mathcal{V}_{\mathrm{et}})$ is an equivalence.

Proof. By the equivalence $\mathcal{U}_{\mathrm{et}} \simeq \mathcal{U}_{\mathrm{et}^{\mathrm{aff}}}$ (Lemma 4.2.4) and Definition 4.2.12 of proper morphisms, we can reduce the problem to the proper base change for modules over sheaves of torsion rings on schemes [SGA4, XII, Théorème 5.1]. □

The proper base change naturally extends to the derived ∞-categories, and we have

Lemma 4.3.2. Under the same assumption with Lemma 4.3.1, the base change morphism $g_{\mathrm{et}}^{*}\, f_{*}^{\mathrm{et}} \to f'^{\mathrm{et}}_{*}\, g'^{*}_{\mathrm{et}}$ of functors between derived ∞-categories $\mathsf{D}_{\infty}(\mathcal{U}'_{\mathrm{et}}, \Lambda) \to \mathsf{D}_{\infty}(\mathcal{V}_{\mathrm{et}}, \Lambda)$ is an equivalence.

4.4. **Direct image functor with proper support.** In §4.6 we will introduce dualizing complexes for derived algebraic spaces. For that, we need the shriek functors $f_!$ and $f^!$, which will be defined in this and the next subsections. Our argument is a simple analogue of that for schemes [SGA4, XVII]. We continue to work over a commutative ring k.

4.4.1. *Open immersions.* Let $j : \mathcal{U} \to \mathcal{V}$ be an open immersion of derived algebraic spaces (Definition 4.2.6). Recall that we defined an open immersion using the more general definition of an open geometric immersion of ∞-topoi (Definition 3.6.2). By the argument in §3.6, associated to j we have two geometric morphisms of ∞-topoi:

$$j_! : \mathcal{U}_{\mathrm{et}} \rightleftarrows \mathcal{V}_{\mathrm{et}} : j^*, \quad j^* : \mathcal{V}_{\mathrm{et}} \rightleftarrows \mathcal{U}_{\mathrm{et}} : j_*.$$

Let $(\mathcal{U}_{\mathrm{et}}, \mathcal{A})$ be a ringed ∞-topos, where $\mathcal{U}_{\mathrm{et}}$ is the étale ∞-topos of the derived algebraic space $\mathcal{U}$. Then we have another ringed ∞-topos $(\mathcal{V}_{\mathrm{et}}, j^*\mathcal{A})$. The above geometric morphisms induce the adjunctions

$$j_! : \mathsf{Mod}_{\mathcal{A}}(\mathcal{U}_{\mathrm{et}}) \rightleftarrows \mathsf{Mod}_{j^*\mathcal{A}}(\mathcal{V}_{\mathrm{et}}) : j^*, \quad j^* : \mathsf{Mod}_{j^*\mathcal{A}}(\mathcal{V}_{\mathrm{et}}) \rightleftarrows \mathsf{Mod}_{\mathcal{A}}(\mathcal{U}_{\mathrm{et}}) : j_*.$$

We call $j_!$ the *extension by zero*, and j^* the *restriction*. We also have a morphism

$$j_! \longrightarrow j_*$$

in $\mathsf{Fun}(\mathsf{Mod}_{\mathcal{A}}(\mathcal{U}_{\mathrm{et}}), \mathsf{Mod}_{j^*\mathcal{A}}(\mathcal{V}_{\mathrm{et}}))$.

4.4.2. *Compactifiable morphism.* Recall that an S-morphism $f : X \to Y$ of schemes over a base scheme S is *S-compactifiable* [SGA4, XVII, Definition 3.2.1] if there exists an S-scheme P which is proper over S and a factorization $f = (X \xrightarrow{j} P \times_S Y \xrightarrow{p_Y} Y)$ with j quasi-finite and separated. If moreover $S = Y$, then f is called *compactifiable.*

Let us define the corresponding notion for algebraic spaces as follows:

Definition. An S-morphism $X \to Y$ of algebraic spaces over a base scheme S is *S-compactifiable* if there exists an algebraic space P over S which is proper (Definition A.1.10) and a factorization $f = (X \xrightarrow{j} P \times_S Y \xrightarrow{p_Y} Y)$ with j quasi-compact (Definition A.1.3), locally quasi-finite (Definition A.1.2) and separated (Definition A.1.5 (2)).

Recalling Definition 4.2.12 of proper morphisms of derived algebraic spaces, we introduce a derived analogue. Let us write the base commutative ring k explicitly for a while.

Definition. A morphism $f : \mathcal{U} \to \mathcal{V}$ of derived algebraic spaces over k is *compactifiable* if there exists a proper derived algebraic space $\mathcal{P}$ (Definition 4.2.12) and a factorization $f = (\mathcal{U} \xrightarrow{j} \mathcal{P} \times_{\mathrm{dSpec}\, k} \mathcal{V} \xrightarrow{p_{\mathcal{V}}} \mathcal{V})$ such that the truncation $\mathrm{Trc}\, f = (\mathrm{Trc}\, \mathcal{U} \xrightarrow{\mathrm{Trc}\, j} \mathrm{Trc}\, \mathcal{U} \times_{\mathrm{Spec}\, k} \mathrm{Trc}\, \mathcal{V} \xrightarrow{p_{\mathrm{Trc}\, \mathcal{V}}} \mathrm{Trc}\, \mathcal{V})$ makes $\mathrm{Trc}\, f$ a compactifiable morphism of algebraic spaces.

Next we introduce an analogue of the category (S) of S-compactifiable morphisms [SGA4, XVII.3.2] in the context of derived algebraic spaces.

Definition. We define $\mathsf{dAS}_k^{\mathrm{cpt}}$ to be the sub-∞-category of dAS_k whose objects are derived algebraic spaces $\mathcal{U}$ over k whose truncations $\mathrm{Trc}\, \mathcal{U}$ are quasi-compact and quasi-separated algebraic spaces over $\mathrm{Spec}\, k$ and whose morphisms are compactifiable morphisms.

Then it is natural to set

Definition. Let $f : \mathcal{U} \to \mathcal{V}$ be a morphism in $\mathsf{dAS}_k^{\mathrm{cpt}}$. A *compactification* of f is a triangle

$$\begin{array}{ccc} \mathcal{U} & \overset{i}{\hookrightarrow} & \overline{\mathcal{U}} \\ {\scriptstyle f}\downarrow & \swarrow {\scriptstyle \overline{f}} & \\ \mathcal{V} & & \end{array}$$

in dAS_k with i an open immersion and $\overline{f}$ proper.

Since our definitions for derived algebraic spaces refer only to the truncated algebraic spaces, the argument in [SGA4, XVII Proposition 3.23] works as it is, and we have

Lemma. Any morphism f in $\mathsf{dAS}_k^{\mathrm{cpt}}$ has a compactification.

Note that an open immersion of derived algebraic spaces lives in $\mathsf{dAS}_k^{\mathrm{cpt}}$. Recall also that we have base change theorems for open immersions (Lemma 3.6.5) and for proper morphisms (Lemma 4.3.2). Then by the construction in [SGA4, XVII.3.3, 5.1] we have

Lemma 4.4.1. Let Λ be a torsion commutative ring. For each morphism $f : \mathcal{U} \to \mathcal{V}$ in the ∞-category $\mathsf{dAS}_k^{\mathrm{cpt}}$, we have a t-exact functor

$$f_! : \mathsf{D}_\infty(\mathcal{U}_{\mathrm{et}}, \Lambda) \longrightarrow \mathsf{D}_\infty(\mathcal{V}_{\mathrm{et}}, \Lambda)$$

which extends $i_! : \mathsf{Mod}(\mathcal{U}_{\mathrm{et}}, \Lambda) \to \mathsf{Mod}(\mathcal{V}_{\mathrm{et}}, \Lambda)$ for open immersion $i : \mathcal{U} \to \mathcal{V}$ in $\mathsf{dAS}_k^{\mathrm{cpt}}$.

4.5. **Extraordinary inverse image functor.** We continue to use the notations in the previous §4.4. By [SGA4, XVIII Théorème 3.1.4] we have

Lemma. Let $f : \mathcal{U} \to \mathcal{V}$ be a morphism in $\mathsf{dAS}_k^{\mathrm{cpt}}$, and Λ be a torsion commutative ring. Then the functor $f_! : \mathsf{D}_\infty(\mathcal{U}_{\mathrm{et}}, \Lambda) \to \mathsf{D}_\infty(\mathcal{V}_{\mathrm{et}}, \Lambda)$ admits a right adjoint t-exact functor

$$f^! : \mathsf{D}^+_\infty(\mathcal{V}_{\mathrm{et}}, \Lambda) \longrightarrow \mathsf{D}^+_\infty(\mathcal{U}_{\mathrm{et}}, \Lambda).$$

We call it the *extraordinary inverse image functor.*

4.6. **Dualizing objects on derived algebraic spaces.** Having introduced functors between sheaves of modules, we can now discuss the dualizing complexes for derived algebraic spaces. Let us apologize that our discussion is not of full generality: we only discuss a rather restricted situation over the base ring k (see Assumption 4.6.1 below). It might be possible to take a more general base scheme as in [LO1, LO2], but we will not pursue this point.

We will define dualizing objects for derived algebraic spaces by some gluing argument, following the discussion for algebraic spaces in [LO1, 3.1].

Let $\mathcal{W}$ be a derived algebraic space over k. By Lemma 4.2.4, we can replace the étale ∞-site $\mathrm{Et}(\mathcal{W}) = (\mathsf{dAS}_{\mathcal{W}}^{\mathrm{et}}, \mathrm{et})$ by the ∞-site $(\mathsf{dAff}_{\mathcal{W}}^{\mathrm{et}}, \mathrm{et}^{\mathrm{aff}})$ consisting only of étale morphisms $U \to \mathcal{W}$ from *affine derived schemes* U, and have an equivalence

$$\mathcal{W}_{\mathrm{et}}|_U \simeq U_{\mathrm{et}^{\mathrm{aff}}}.$$

Here the left hand side is the localized ∞-topos (Fact 1.8.6).

Hereafter we assume the following conditions.

Assumption 4.6.1.
- The base ring k is a finite field or a separably closed field.
- The commutative ring Λ is a torsion noetherian ring annihilated by an integer invertible in k.

- The derived algebraic space $\mathcal{W}$ is separated (Definition 4.2.7) and of finite presentation as a geometric derived stack (Definition 2.2.26).

Then we can further replace the underlying-∞-category $\mathsf{dAff}^{\mathrm{et}}_{\mathcal{W}}$ of the étale ∞-site by the full sub-∞-category $\mathsf{dAff}^{\mathrm{et,fp}}_{\mathcal{W}}$ spanned by affine derived schemes of finite presentation (Definition 2.2.6). By this observation, we regard

$$U_{\mathrm{et}^{\mathrm{aff}}} = \mathsf{Sh}(\mathsf{dAff}^{\mathrm{et,fp}}_{\mathcal{W}}, \mathrm{et})$$

in the following discussion.

A complex of k-module can be regarded as an object of the derived ∞-category $\mathsf{D}_\infty((\mathrm{dSpec}\,k)_{\mathrm{et}^{\mathrm{aff}}}, \Lambda)$ of étale sheaves Λ-modules over $\mathrm{dSpec}\,k$. Here $\mathrm{dSpec}\,k$ is seen as a derived algebraic space. We fine the dualizing complex over $\mathrm{dSpec}\,k$ to be

$$\Omega_k := \Lambda \in \mathsf{D}_\infty((\mathrm{dSpec}\,k)_{\mathrm{et}^{\mathrm{aff}}}, \Lambda).$$

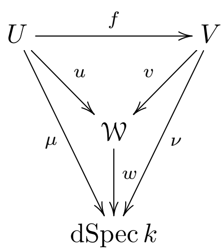
Recalling the extraordinary inverse image functor $\mu^!$ in §4.5, we introduce

Definition. Let $u : U \to \mathcal{W}$ be an object of $\mathsf{dAff}^{\mathrm{et}}_{\mathcal{W}}$, i.e., an étale morphism from an affine derived scheme U over k. Let also $\mu : U \to \mathrm{dSpec}\,k$ be the structure morphism. We define the *relative dualizing object* Ω_u to be

$$\Omega_u := \mu^! \Omega_k \in \mathsf{D}_\infty(U_{\mathrm{et}^{\mathrm{aff}}}, \Lambda).$$

Lemma 4.6.2. The above construction $(u : U \to \mathcal{W}) \mapsto \Omega_u$ is functorial in the ∞-topos $\mathcal{W}_{\mathrm{et}^{\mathrm{aff}}}$. In other words, for any morphism $f : U \to V$ in $\mathcal{W}_{\mathrm{et}^{\mathrm{aff}}}$, we have a functorial isomorphism $f^*\Omega_v \simeq \Omega_u$.

Proof. In the square



f is étale since u and v are so. Thus we have the inverse image functor $f^* : \mathsf{D}_\infty(V_{\mathrm{et}^{\mathrm{aff}}}, \Lambda) \to \mathsf{D}_\infty(U_{\mathrm{et}^{\mathrm{aff}}}, \Lambda)$, which is the desired one. □

Recall the restriction of sheaves (Notation 3.1.6).

Proposition 4.6.3. There exists an object $\Omega_w \in \mathsf{D}_\infty(\mathcal{W}_{\mathrm{et}}, \Lambda)$, uniquely up to contractible ambiguity, such that $\Omega_w|_U = \Omega_u$.

Proof. By the discussion in [SGA4$\frac{1}{2}$, Th. finitude, Théorème 4.3] we have $\mathscr{H}om_\Lambda(\Omega_u, \Omega_u) = \Lambda$, which implies $\mathrm{Ext}^i_\Lambda(\Omega_u, \Omega_u) = 0$ for any $i < 0$. Then the gluing lemma (Fact 3.7.9) gives the consequence. □

The object Ω_w satisfies the following properties.

Lemma 4.6.4. (1) Ω_w is of finite injective dimension. In other words, $\pi_n \Omega_w = 0$ for $n \gg 0$.
(2) For every $\mathcal{M} \in \mathsf{D}_\infty(\mathcal{W}_{\mathrm{et}}, \Lambda)$, the canonical map

$$\mathrm{Map}_{\mathsf{D}_\infty(\mathcal{W}_{\mathrm{et}},\Lambda)}(\mathcal{M}, \Lambda) \longrightarrow \mathrm{Map}_{\mathsf{D}_\infty(\mathcal{W}_{\mathrm{et}},\Lambda)}(\mathcal{M} \otimes_\Lambda \Omega_w, \Omega_w)$$

is an equivalence in $\mathcal{H}$.

Proof. (1) is the consequence of Assumption 4.6.1. (2) is by construction. □

Following the terminology in [Lur14, Definition 4.2.5], we name

Notation 4.6.5. We call $\Omega_w \in \mathsf{D}_\infty(\mathcal{W}_{\mathrm{et}}, \Lambda)$ the *dualizing object*.

In the later §5.5 we will discuss lisse-étale sheaves on derived stacks. For that, we need functionality of dualizing objects with respect to smooth morphisms. Now the Tate twist comes into play as in the cases of schemes and of algebraic spaces. Let us set a notation for the Tate twist.

Notation 4.6.6. Under Assumption 4.6.1 on k and Λ, for $\mathcal{M} \in \mathsf{D}_\infty(\mathcal{W}_{\mathrm{et}}, \Lambda)$ and $d \in \mathbb{Z}$, we denote by $\mathcal{M}(d)$ the d-th Tate twist and set

$$\mathcal{M}\langle d \rangle := \mathcal{M}(d)[2d].$$

Now the proof of [LO1, 3.1.2 Lemma] works for derived algebraic spaces, and we have

Lemma 4.6.7. Let $\mathcal{W}_1$ and $\mathcal{W}_2$ be derived algebraic spaces over k which are separated (Definition 4.2.7) and finitely presented (Definition 2.2.21), and let $f : \mathcal{W}_1 \to \mathcal{W}_2$ be a smooth morphism of relative dimension d (Definition 2.2.23). We denote by Ω_i the dualizing objects of $\mathcal{W}_i$ $(i = 1, 2)$. Then we have an equivalence

$$f^*\Omega_2 \simeq \Omega_1\langle -d\rangle.$$

5. Lisse-étale sheaves on derived stacks

The purpose of this section is to introduce the lisse-étale site on a derived stack of certain type and build a theory of constructible sheaves with finite coefficient on the lisse-étale site.

Let us recall the situation in the ordinary setting. For an algebraic stack X, we consider the lisse-étale site whose underlying category consists of algebraic spaces smooth over X and whose Grothendieck topology is given by the étale coverings. In [LM] the theory of derived category of constructible sheaves on this lisse-étale site is developed. As explained in [O1], some arguments in [LM] are not correct due to the non-functoriality of the lisse-étale site. Using cohomological descent those errors are fixed in [O1]. Based on this correction, the theory of derived functors of $\overline{\mathbb{Q}}_\ell$-sheaves on algebraic stacks is constructed in [LO1, LO2].

As in the previous §4, we work over a fixed base commutative ring k unless otherwise stated, and suppress the notation of k.

5.1. The lisse-étale site.
The lisse-étale site for algebraic stacks is introduced in [LM, Chap. 12] in order to build a reasonable theory of sheaves on algebraic stacks. Its definition is a mixture of étale and smooth morphisms, reflecting Definition A.2.3 of algebraic stacks which uses both étale and smooth morphisms. Our definition of the lisse-étale ∞-site for a derived stack will be a simple analogue of this lisse-étale site.

We begin with somewhat similar argument to §4.2. For a derived stack $\mathcal{X}$, we have the over-∞-category $\mathsf{dSt}_{/\mathcal{X}}$, which can be regarded as the ∞-category of pairs $(\mathcal{Y}, y)$ consisting of $\mathcal{Y} \in \mathsf{dSt}$ and $y : \mathcal{Y} \to \mathcal{X}$ a morphism in dSt. We denote by $\mathsf{dAS}_{/\mathcal{X}}$ the full sub-∞-category of $\mathsf{dSt}_{/\mathcal{X}}$ spanned by those pairs $(\mathcal{U}, u)$ with $\mathcal{U}$ a derived algebraic space and $u : \mathcal{U} \to \mathcal{X}$ a morphism in dSt.

Definition. Let $\mathcal{X}$ be a geometric derived stack.

1. The *lisse-étale ∞-site* of $\mathcal{X}$ is the ∞-site $\mathrm{LE}(\mathcal{X}) := (\mathsf{dAS}^{\mathrm{lis}}_{\mathcal{X}}, \text{lis-et})$ consisting of
   - The full sub-∞-category $\mathsf{dAS}^{\mathrm{lis}}_{\mathcal{X}}$ of $\mathsf{dAS}_{/\mathcal{X}}$ spanned those $(\mathcal{U}, u)$ with $u : \mathcal{U} \to \mathcal{X}$ a smooth morphism of derived stacks (Definition 2.2.21).
   - The Grothendieck topology lis-et, called the *lisse-étale topology*, on $\mathsf{dAS}^{\mathrm{lis}}_{\mathcal{U}}$ whose set $\mathrm{Cov}_{\text{lis-et}}(\mathcal{U})$ of covering sieves of $\mathcal{U} \in \mathsf{dAS}^{\mathrm{lis}}_{\mathcal{U}}$ consists of families $\{\mathcal{U}_i \to \mathcal{U}\}_{i \in I}$ such that each $\mathcal{U}_i \to \mathcal{U}$ is an étale morphism derived stacks (Definition 2.2.21), and the induced $\coprod_{i \in I} \mathcal{U}_i \to \mathcal{U}$ is an epimorphism of derived stacks (Definition 2.2.11).
2. We denote the associated ∞-topos (Fact 1.7.4) by $\mathcal{X}_{\text{lis-et}} := \mathsf{Sh}(\mathrm{LE}(\mathcal{X}))$ and call it the *lisse-étale ∞-topos* on $\mathcal{X}$.

As in the non-derived case [LM, Lemma (12.1.2)] we have

Lemma. $\mathcal{X}_{\text{lis-et}}$ is equivalent to the ∞-topos arising from the ∞-site $(\mathsf{dAS}^{\mathrm{lis}}_{\mathcal{X}}, \text{lis-et}')$ where lis-et$'$ is the Grothendieck topology whose covering sheaves consist only of *finite* families of étale morphisms.

Proof. The same argument as in the proof of [LM, Lemma (12.1.2)] works since dSt is quasi-compact (Fact 2.2.10). □

We also have the following obvious statement.

Lemma 5.1.1. For a derived algebraic space $\mathcal{U}$, the identity functor $\mathsf{dAS}^{\mathrm{et}}_{\mathcal{U}} \ni (\mathcal{U}', u') \mapsto (\mathcal{U}', u') \in \mathsf{dAS}^{\text{lis-et}}_{\mathcal{U}}$ gives a continuous functor $\mathrm{Et}(\mathcal{U}) \to \mathrm{LE}(\mathcal{U})$ of ∞-sites. It induces an equivalence of ∞-topoi

$$\varepsilon : \mathcal{U}_{\text{lis-et}} \longrightarrow \mathcal{U}_{\mathrm{et}}.$$

We have a similar statement to Lemma 4.2.4.

Lemma. Let $\mathcal{X}$ be a geometric derived stack, and $\mathrm{LE}^{\mathrm{aff}}(\mathcal{X}) := (\mathsf{dAff}^{\mathrm{et}}_{\mathcal{X}}, \text{lis-et}^{\mathrm{aff}})$ be the ∞-site consisting of

- The full-∞-subcategory $\mathsf{dAff}^{\mathrm{et}}_{\mathcal{X}}$ of $\mathsf{dAff}_{/\mathcal{X}}$ spanned by affine derived $\mathcal{X}$-schemes (U, u) with $u : U \to \mathcal{X}$ a smooth morphism of derived stacks. Here $\mathsf{dAff}_{/\mathcal{X}}$ denotes the full sub-∞-category of $\mathsf{dSt}_{/\mathcal{X}}$ spanned by affine derived schemes over $\mathcal{X}$.
- The Grothendieck topology lis-et$^{\mathrm{aff}}$ where a covering sieve is defined to be a covering sieve in et on $\mathsf{dAS}^{\mathrm{lis}}_{\mathcal{X}}$.

Then the associated ∞-topos $\mathcal{X}_{\text{lis-et}^{\text{aff}}} := \mathsf{Sh}(\mathrm{LE}^{\text{aff}}(\mathcal{X}))$ is equivalent to $\mathcal{X}_{\text{lis-et}}$

We have the following relationship between the lisse-étale ∞-topos introduced above and the lisse-étale topos on an algebraic stack in the ordinary sense (Definition A.2.3). Let X be an algebraic stack. Then applying the functor $\iota = \mathrm{Dex} \circ a$ (Definition 2.2.29) we have a derived stack $\iota(X)$, which is 1-geometric by Fact 2.1.9. Thus we have the lisse-étale ∞-topos $\iota(X)_{\text{lis-et}}$. Then by Lemma 1.8.2, we have a topos $\iota(X)^{\text{cl}}_{\text{lis-et}}$ in the ordinary sense.

Lemma 5.1.2. For an algebraic stack X, the topos $\iota(X)^{\text{cl}}_{\text{lis-et}}$ is equivalent to the lisse-étale topos $X_{\text{lis-et}}$ (Definition A.3.1).

Proof. Since $\iota(X)$ is 1-geometric, the underlying ∞-category $\mathsf{dAS}^{\text{lis}}_{\iota(X)}$ of $\mathrm{LE}(\mathcal{X})$ is equivalent to the nerve of the underlying category of the lisse-étale site on X in Definition A.3.1. It implies the conclusion. □

In the remaining part of this subsection, we give a preliminary discussion on functors of lisse-étale sheaves. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of geometric derived stacks. It gives rise to a continuous functor

$$\mathrm{LE}(\mathcal{Y}) \longrightarrow \mathrm{LE}(\mathcal{X}), \quad U \longmapsto U \times_{\mathcal{Y}} \mathcal{X}$$

of ∞-sites. In fact, the morphism $U \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X}$ is smooth by [TVe2, Lemma 2.2.3.1 (2)] and if $\{U_i \to U\}_{i \in I}$ is an étale covering then $\{U_i \times_{\mathcal{Y}} \mathcal{X} \to U \times_{\mathcal{Y}} \mathcal{X}\}_{i \in I}$ is also an étale covering. Thus $U \times_{\mathcal{Y}} \mathcal{X}$ belongs to $\mathrm{LE}(\mathcal{X})$.

Lemma 5.1.3. For a morphism $f : \mathcal{X} \to \mathcal{Y}$ of n-geometric derived stacks, we have a pair of functors

$$f^{-1}_{\text{lis-et}} : \mathcal{Y}_{\text{lis-et}} \longrightarrow \mathcal{X}_{\text{lis-et}}, \quad f^{\text{lis-et}}_{*} : \mathcal{X}_{\text{lis-et}} \longrightarrow \mathcal{Y}_{\text{lis-et}}$$

of ∞-topoi by the following construction.

(1) For $\mathcal{F} \in \mathcal{X}_{\text{lis-et}}$, we define $f_* \mathcal{F} \in \mathcal{Y}_{\text{lis-et}}$ by

$$(f^{\text{lis-et}}_{*} \mathcal{F})(U) := \mathcal{F}(U \times_{\mathcal{Y}} \mathcal{X}).$$

(2) For $\mathcal{G} \in \mathcal{Y}_{\text{lis-et}}$, we define $f^{-1}_{\text{lis-et}} \mathcal{G} \in \mathcal{Y}_{\text{lis-et}}$ to be the sheafification of the presheaf given by

$$\mathrm{LE}(\mathcal{X}) \ni V \longmapsto \varinjlim_{V \to U} \mathcal{F}(U).$$

Here the colimit is taken in the sub-∞-category of the over-∞-category $\mathsf{dSt}_{/f}$ spanned by morphisms $V \to U$ over f with $U \in \mathrm{LE}(\mathcal{Y})$. In other words, we are considering a square

$$\begin{array}{ccc} V & \longrightarrow & U \\ \downarrow & & \downarrow \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \end{array} \tag{5.1.1}$$

where vertical morphisms are smooth.

As in the ordinary case indicated by [O1, §3.3], the functor f^{-1} is not left exact since the colimit in its definition is not filtered (recall Definition B.10.1 of left exactness). In other words, the pair (f^{-1}, f_*) does not give a geometric morphism of ∞-topoi (Definition 3.1.1). However, if f is smooth, then f^{-1} is just the restriction functor as the diagram (5.1.1) suggests, so that it is exact. We take the same strategy as in [O1] to remedy this non-exactness problem, which will be explained in the following subsections.

Let us close this subsection by introducing a terminology for sheaves over $\mathcal{X}_{\text{lis-et}}$.

Notation. Let $\mathcal{X}$ be a geometric derived stack and C be an ∞-category admitting small limits. An object of $\mathsf{Shv}_{\mathsf{C}}(\mathcal{X}_{\text{lis-et}})$ will be called a *lisse-étale sheaf valued in* C *on* $\mathcal{X}$.

Let us close this subsection by introducing the lisse-étale ∞-topos for *locally geometric* derived stacks (Definition 2.2.18). Recall that a locally geometric derived stack $\mathcal{X}$ can be expressed as a colimit $\varinjlim_{i \in I} \mathcal{X}_i$ of a filtered family of geometric derived stacks $\mathcal{X}_i$.

Definition 5.1.4. Let $\mathcal{X} \simeq \varinjlim_{i \in I} \mathcal{X}_i$ be a locally geometric derived stack with $\{\mathcal{X}_i\}$ a filtered family of geometric derived stacks. Then we define the *lisse-étale ∞-topos* $\mathcal{X}_{\text{lis-et}}$ of $\mathcal{X}$ to be the colimit

$$\mathcal{X}_{\text{lis-et}} := \varinjlim_{i \in I} (\mathcal{X}_i)_{\text{lis-et}}$$

of the lisse-étale ∞-topoi of $\mathcal{X}_i$. Here we take the colimit in the ∞-category RTop of ∞-categories, which admits small colimits (Fact 3.1.4).

5.2. **Lisse-étale sheaves on derived stacks.** In this subsection we introduce the notion of lisse-étale sheaves of modules on geometric derived stacks. As before we work on a fixed commutative ring k.

5.2.1. *Notations on lisse-étale sheaves.* Throughout this section we will use various notions of sheaves on ringed ∞-topoi discussed in §3. So let us recollect some of them as a preliminary.

Fix a geometric derived stack $\mathcal{X}$ over k. Then applying to $\mathsf{T} = \mathcal{X}_{\text{lis-et}}$ the definitions in §3, we have the following ∞-categories of lisse-étale sheaves.

Notation. (1) We call $\mathsf{Shv}_{\mathsf{Com}}(\mathcal{X}_{\text{lis-et}})$ the ∞-category of *lisse-étale sheaves of commutative rings* on $\mathcal{X}$.
(2) For an $\mathcal{A} \in \mathsf{Shv}_{\mathsf{Com}}(\mathcal{X}_{\text{lis-et}})$, we call

$$\mathsf{Mod}_{\mathcal{A}}(\mathcal{X}_{\text{lis-et}})$$

the ∞-category of *lisse-étale sheaves of* $\mathcal{A}$*-modules* on $\mathcal{X}$. We also call the full sub-∞-category

$$\mathsf{Mod}^{\text{stab}}_{\mathcal{A}}(\mathcal{X}_{\text{lis-et}}) \subset \mathsf{Mod}_{\mathcal{A}}(\mathcal{X}_{\text{lis-et}})$$

the ∞-category of *lisse-étale sheaves of stable* $\mathcal{A}$*-modules on* $\mathcal{X}$.

For an $\mathcal{A} \in \mathsf{Shv}_{\mathsf{Com}}(\mathcal{X}_{\text{lis-et}})$, the homotopy category

$$\mathrm{Mod}_{\mathcal{A}}(\mathcal{X}_{\text{lis-et}}) := \mathrm{h}\,\mathsf{Mod}_{\mathcal{A}}(\mathcal{X}_{\text{lis-et}})$$

is a Grothendieck abelian category (Proposition 3.5.1). So we can consider the associated derived ∞-category $\mathsf{D}_{\infty}(\mathrm{Mod}_{\mathcal{A}}(\mathcal{X}_{\text{lis-et}}))$ (§3.5.2). Recalling the equivalence in Proposition 3.5.3, we set

Notation 5.2.1. For an $\mathcal{A} \in \mathsf{Shv}_{\mathsf{Com}}(\mathcal{X}_{\text{lis-et}})$, we denote

$$\mathsf{D}_{\infty}(\mathcal{X}_{\text{lis-et}}, \mathcal{A}) := \mathsf{D}_{\infty}(\mathrm{Mod}_{\mathcal{A}}(\mathcal{X}_{\text{lis-et}})) \simeq \mathsf{Mod}^{\text{stab}}_{\mathcal{A}}(\mathcal{X}_{\text{lis-et}})$$

and call it the *derived* ∞*-category of lisse-étale sheaves of* $\mathcal{A}$*-modules* on $\mathcal{X}$.

In the rest of this part, we introduce a derived analogue of *flat* sheaves of commutative rings [LM, Chap. 12, (12.7)]. In fact, the contents will not be used essentially in our main argument, so the readers may skip it.

In order to introduce the flatness, let us consider the following situation: Let $\mathcal{X}$ be a geometric derived stack. Then for each object $(\mathcal{U}, u : \mathcal{U} \to \mathcal{X})$ of the underlying ∞-category $\mathsf{dAS}^{\text{lis}}_{\mathcal{X}}$ of $\mathrm{LE}(\mathcal{X})$, we have the pair of functors

$$u^{-1}_{\text{lis-et}} : \mathcal{X}_{\text{lis-et}} \longrightarrow \mathcal{U}_{\text{lis-et}}, \quad u^{\text{lis-et}}_{*} : \mathcal{U}_{\text{lis-et}} \longrightarrow \mathcal{X}_{\text{lis-et}}$$

of ∞-topos by Lemma 5.1.3.

On the other hand, we have the equivalence $\varepsilon : \mathcal{U}_{\text{lis-et}} \xrightarrow{\sim} \mathcal{U}_{\text{et}}$ of ∞-topoi by Lemma 5.1.1. Thus we can introduce

Notation 5.2.2. For $\mathcal{F} \in \mathcal{X}_{\text{lis-et}}$ and $(\mathcal{U}, u) \in \mathsf{dAS}^{\text{lis}}_{\mathcal{X}}$, we denote

$$\mathcal{F}_{(\mathcal{U},u)} := \varepsilon u^{-1}_{\text{lis-et}}(\mathcal{F}) \in \mathcal{U}_{\text{et}}.$$

If no confusion would occur, then we simply denote $\mathcal{F}_U := \mathcal{F}_{(U,u)}$.

Next, let $f : \mathcal{U} \to \mathcal{V}$ be a morphism in the underlying ∞-category $\mathsf{dAS}^{\text{lis}}_{\mathcal{X}}$ of $\mathrm{LE}(\mathcal{X})$. By Lemma 5.1.3, we have a functor $f^{-1} : \mathcal{V}_{\text{lis-et}} \to \mathcal{U}_{\text{lis-et}}$. Using the equivalence ε in Lemma 5.1.1, we have a functor $\mathcal{V}_{\text{et}} \to \mathcal{U}_{\text{et}}$. Abusing symbols, we denote it by

$$f^{-1} : \mathcal{V}_{\text{et}} \longrightarrow \mathcal{U}_{\text{et}}.$$

Then, for $\mathcal{F} \in \mathcal{X}_{\text{lis-et}}$, we have a morphism $f^{-1}\mathcal{F}_{\mathcal{V}} \to \mathcal{F}_{\mathcal{U}}$ in $\mathcal{U}_{\text{et}}$ since $\mathcal{F}$ is a sheaf on $\mathsf{dAS}^{\text{lis}}_{\mathcal{X}}$.

Note that we can replace the ∞-category $\mathcal{X}_{\text{lis-et}} \simeq \mathsf{Shv}_{\mathsf{S}}(\mathcal{X}_{\text{lis-et}})$ by the ∞-category $\mathsf{Shv}_{\mathsf{Com}}(\mathcal{X}_{\text{lis-et}})$ of lisse-étale sheaves of commutative rings in the argument so far. We summarize it in

Notation 5.2.3. Let $\mathcal{A} \in \mathsf{Shv}_{\mathsf{Com}}(\mathcal{X}_{\text{lis-et}})$ be a lisse-étale sheaf of commutative rings on $\mathcal{X}$, and $f : (\mathcal{U}, u) \to (\mathcal{V}, v)$ be a morphism in $\mathsf{dAS}^{\text{lis}}_{\mathcal{X}}$. We denote

$$\mathcal{A}_{\mathcal{U}} = \mathcal{A}_{(\mathcal{U},u)} := \varepsilon u^{-1}(\mathcal{A}) \in \mathsf{Shv}_{\mathsf{Com}}(\mathcal{U}_{\text{et}})$$

and call it the *restriction of* $\mathcal{A}$ *on* $(\mathcal{U}, u)$. Then we have a natural morphism $f^{-1}(\mathcal{A}_{\mathcal{V}}) \to \mathcal{A}_{\mathcal{U}}$ in the ∞-category $\mathsf{Shv}_{\mathsf{Com}}(\mathcal{U}_{\text{et}})$ of étale sheaves of commutative rings on the derived algebraic space $\mathcal{U}$.

We use a similar notation $\mathcal{M}_U \in \mathsf{Mod}_{\mathcal{A}_U}(\mathcal{U}_{\text{et}})$ for $\mathcal{M} \in \mathsf{Mod}_{\mathcal{A}}(\mathcal{X}_{\text{lis-et}})$.

Now we introduce a derived analogue of the flatness of a sheaf of commutative rings [O1, Definition 3.7].

Definition 5.2.4. Let $\mathcal{X}$ be a geometric derived stack, and $\mathcal{A}$ be a lisse-étale sheaf of commutative rings on $\mathcal{X}$. Then $\mathcal{A}$ is *flat* if for any morphism $f : \mathcal{U} \to \mathcal{V}$ in $\mathsf{dAS}^{\text{lis}}_{\mathcal{X}}$, the morphism $f^{-1}(\mathcal{A}_V) \to \mathcal{A}_U$ in $\mathsf{Shv}_{\mathsf{Com}}(\mathcal{U}_{\text{et}})$ is faithfully flat (in the ordinary sense).

5.2.2. *Cartesian sheaves.* In [LM, Définition (12.3), (12.7.3)], the notion of *cartesian sheaves* on algebraic stacks is introduced and used to define quasi-coherent sheaves, derived categories of quasi-coherent sheaves and derived functors between them. Let us briefly recall its definition.

Definition 5.2.5 ([LM, Définition (12.7.3)])**.** Let X be an algebraic stack (Definition A.2.3) and $\mathcal{A}$ be a flat sheaf of commutative rings on the lisse-étale topos $X_{\text{lis-et}}$ (Definition A.3.1). A sheaf $\mathcal{M}$ of $\mathcal{A}$-modules on $X_{\text{lis-et}}$ is *cartesian* if for any morphism $f : U \to V$ in the lisse-étale site of X the natural morphism $\mathcal{A}_U \otimes_{f^{-1}(\mathcal{A}_V)} f^{-1}(\mathcal{M}_V) \to \mathcal{M}_U$ of sheaves on U_{et} is an isomorphism.

The essence of cartesian sheaves is shown in the following fact.

Fact 5.2.6 ([O1, Lemma 3.8])**.** Let X and $\mathcal{A}$ be as in Definition 5.2.5. A sheaf of $\mathcal{A}$-modules $\mathcal{M}$ on $X_{\text{lis-et}}$ is cartesian if and only if for every *smooth* morphism $f : U \to V$ in the lisse-étale site of X the natural morphism $\mathcal{A}_U \otimes_{f^{-1}(\mathcal{A}_V)} f^{-1}(\mathcal{M}_V) \to \mathcal{M}_U$ is an isomorphism.

Thus a cartesian sheaf is a sheaf of modules which is totally characterized by the behavior under smooth morphisms.

Now let us introduce cartesian lisse-étale sheaves. We continue to use the notations in the previous parts, so that $\mathcal{U}$ and $\mathcal{V}$ denote derived algebraic spaces in the underlying ∞-category $\mathsf{dAS}^{\text{lis}}_{\mathcal{X}}$ of the lisse-étale ∞-site $\mathrm{LE}(\mathcal{X})$ of a derived stack $\mathcal{X}$. Recall also Notation 5.2.3 on the restriction of sheaves.

Definition 5.2.7. Let $\mathcal{X}$ be a geometric derived stack and $\mathcal{A}$ be a flat lisse-étale sheaf of commutative rings on $\mathcal{X}$. An $\mathcal{A}$-modules $\mathcal{M} \in \mathsf{Mod}_{\mathcal{A}}(\mathcal{X}_{\text{lis-et}})$ is called *cartesian* if for any morphism $f : \mathcal{U} \to \mathcal{V}$ in $\mathsf{dAS}^{\text{lis}}_{\mathcal{X}}$, the morphism

$$\mathcal{A}_{\mathcal{U}} \otimes_{f^{-1}(\mathcal{A}_{\mathcal{V}})} f^{-1}(\mathcal{M}_{\mathcal{V}}) \longrightarrow \mathcal{M}_{\mathcal{U}}$$

is an equivalence in $\mathsf{Mod}_{\mathcal{A}_{\mathcal{U}}}(\mathcal{U}_{\text{et}})$. We denote by

$$\mathsf{Mod}^{\text{cart}}_{\mathcal{A}}(\mathcal{X}_{\text{lis-et}}) \subset \mathsf{Mod}_{\mathcal{A}}(\mathcal{X}_{\text{lis-et}})$$

the full sub-∞-category spanned by cartesian sheaves.

We have an analogue of Fact 5.2.6.

Lemma 5.2.8. Let $\mathcal{X}$ be a geometric derived stack, $\mathcal{A} \in \mathsf{Shv}_{\mathsf{Com}}(\mathcal{X}_{\text{lis-et}})$ be flat, and $\mathcal{M} \in \mathsf{Mod}_{\mathcal{A}}(\mathcal{X}_{\text{lis-et}})$. Then $\mathcal{M}$ is cartesian if and only if for every morphism $f : \mathcal{U} \to \mathcal{V}$ in $\mathsf{dAS}^{\text{lis}}_{\mathcal{X}}$ the natural morphism $\mathcal{A}_{\mathcal{U}} \otimes_{f^{-1}(\mathcal{A}_{\mathcal{V}})} f^{-1}(\mathcal{M}_{\mathcal{V}}) \to \mathcal{M}_{\mathcal{U}}$ is an equivalence in $\mathsf{Mod}_{\mathcal{A}}(\mathcal{U}_{\text{et}})$.

Proof. We follow the proof of [O1, Lemma 3.8]. By the definition of cartesian sheaf it is enough to show the "if" part. Let $\mathcal{X}$ be an n-geometric derived stack. The proof is by induction on n. Assume $n = 1$. Take a 1-atlas $\{X_i\}_{i \in I}$ of $\mathcal{X}$ and consider the 1-smooth effective epimorphism $X' := \coprod_{i \in I} X \twoheadrightarrow \mathcal{X}$. Given a morphism $f : \mathcal{U} \to \mathcal{V}$ in $\mathsf{dAS}^{\text{lis}}_{\mathcal{X}}$, we set $\mathcal{V}' := \mathcal{V} \times_{\mathcal{X}} X'$ and $\mathcal{U}' := \mathcal{U} \times_{\mathcal{X}} X'$. Then we have a pullback square

$$\begin{array}{ccc} \mathcal{U}' & \xrightarrow{f'} & \mathcal{V}' \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ \mathcal{U} & \xrightarrow[f]{} & \mathcal{V} \end{array}$$

in $\mathsf{dAS}_{/\mathcal{X}}$. We also find that u and v are smooth and surjective and that the composition $x : \mathcal{U}' \xrightarrow{f'} \mathcal{V}' \to \mathcal{X}$ is smooth. In order to prove that $\mathcal{A}_{\mathcal{U}} \otimes_{f^{-1}(\mathcal{A}_{\mathcal{V}})} f^{-1}(\mathcal{M}_{\mathcal{V}}) \to \mathcal{M}_{\mathcal{U}}$ is an equivalence, it is enough to show that

$$i : \mathcal{A}_{\mathcal{U}'} \otimes_{u^{-1}f^{-1}(\mathcal{A}_V)} u^{-1}f^{-1}(\mathcal{M}_V) \longrightarrow \mathcal{M}_{\mathcal{U}'}$$

is an equivalence since u is smooth and surjective. Hereafter let us suppress the change of ring, and denote $i : u^{-1}f^{-1}(\mathcal{M}_V) \to \mathcal{M}_{\mathcal{U}'}$. By the above square and the assumption we find $u^{-1}f^{-1}(\mathcal{M}_{\mathcal{V}}) \simeq {f'}^{-1}(\mathcal{M}_{\mathcal{V}'})$, and by the smoothness of u and the assumption we find $u^{-1}(\mathcal{M}_{\mathcal{U}}) \simeq \mathcal{M}_{\mathcal{U}'}$. Thus we have $i : f^{-1}(\mathcal{M}_{\mathcal{V}'}) \to \mathcal{M}_{\mathcal{U}'}$. On the other hand, i is equivalent to the pullback by x of $\mathrm{id}_{\mathcal{F}_X}$. Then by the smoothness of x and the assumption we find that i is an equivalence. □

As a corollary, we have

Corollary 5.2.9. The homotopy category

$$\mathrm{h}\,\mathsf{Mod}^{\text{cart}}_{\mathcal{A}}(\mathcal{X}_{\text{lis-et}}) \subset \mathrm{h}\,\mathsf{Mod}_{\mathcal{A}}(\mathcal{X}_{\text{lis-et}})$$

is a Serre subcategory.

Now we introduce notations on the derived ∞-category of lisse-étale sheaves with cartesian homotopy groups.

Notation 5.2.10. Let $\mathcal{X}$ be a geometric derived stack and $\mathcal{A} \in \mathsf{Shv}_{\mathsf{Com}}(\mathcal{X}_{\text{lis-et}})$ be flat. We define

$$\mathsf{D}^{\text{cart}}_{\infty}(\mathcal{X}_{\text{lis-et}}, \mathcal{A}) \subset \mathsf{D}_{\infty}(\mathcal{X}_{\text{lis-et}}, \mathcal{A}) \simeq \mathsf{Mod}^{\text{stab}}_{\mathcal{A}}(\mathcal{X}_{\text{lis-et}})$$

the full sub-∞-category of the derived ∞-category spanned by those objects $\mathcal{M}$ whose homotopy group $\pi_j(\mathcal{M}) \in \mathsf{Mod}_{\mathcal{A}}(\mathcal{X}_{\text{lis-et}})$ is cartesian for all $j \in \mathbb{Z}$.

Recall that for a stale ∞-category C with a t-structure we have the sub-∞-categories C^+, C^- and C^b of left bounded, right bounded and bounded objects (Definition D.2.3).

Notation. Let $\mathcal{X}$ be a geometric derived stack, and $\mathcal{A} \in \mathsf{Shv}_{\mathsf{Com}}(\mathcal{X}_{\text{lis-et}})$ be flat. We denote by

$$\mathsf{D}^{\text{cart},*}_{\infty}(\mathcal{X}_{\text{lis-et}}, \mathcal{A}) \subset \mathsf{D}^{\text{cart}}_{\infty}(\mathcal{X}_{\text{lis-et}}, \mathcal{A}) \quad (* \in \{+, -, b\})$$

the sub-∞-categories of left bounded, right bounded and bounded objects respectively.

5.3. **Lisse-étale ∞-topos and simplicial étale ∞-topos.** As mentioned in the beginning of §4, the lisse-étale topos on an algebraic stack can be described by the étale topos on a simplicial algebraic space. See [O1] and [O2, §9.2] for the detailed explanation of the non-derived case. In this subsection we give its derived analogue.

Recall Fact 4.1.2. For an n-geometric derived stack $\mathcal{X}$, there exists a family $\{\mathcal{U}_i\}_{i \in I}$ of derived algebraic spaces such that each of them is equipped with an $(n-1)$-smooth morphism $\mathcal{U}_i \to \mathcal{X}$ and $\mathcal{X} \simeq \varinjlim_j \mathcal{X}_j$ with $\mathcal{X}_0 := \coprod_{i \in I} \mathcal{U}_i$ and $\mathcal{X}_j := \mathcal{X}_0 \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} \mathcal{X}_0$ for $j \in \mathbb{Z}_{\geq 1}$ (j-times fiber product). These $\mathcal{X}_j$'s gives rise to a simplicial object $\mathcal{X}_\bullet$ in $\mathsf{dAS}^{\text{lis}}_{\mathcal{X}}$, the underlying ∞-category of the lisse étale ∞-site $\operatorname{LE}(\mathcal{X})$. Indeed, $\mathcal{X}_n$ is a derived algebraic space for each $n \in \mathbb{N}$. We also note that the morphism $\mathcal{X}_n \to \mathcal{X}_m$ is smooth for each $[m] \to [n]$ in $\boldsymbol{\Delta}$.

Definition 5.3.1. We call the simplicial object $\mathcal{X}_\bullet$ in $\mathsf{dAS}^{\text{lis}}_{\mathcal{X}}$ a *smooth presentation of* $\mathcal{X}$, and denote this situation by

$$\mathcal{X}_\bullet \to \mathcal{X}.$$

Note that a smooth presentation is nothing but the (0-)coskeleton of the surjection $\mathcal{X}_1 \to \mathcal{X}$ in the sense of [O2, 9.2.1], and is also equivalent to the hypercovering of $\mathcal{X}$ in the sense of Definition C.3.2.

Restricting to the subcategory $\boldsymbol{\Delta}^{\text{str}} \subset \boldsymbol{\Delta}$ (Definition 3.7.3), we have a strictly simplicial object $\mathcal{X}^{\text{str}}_\bullet$ of derived algebraic spaces.

Since each morphism in $\mathcal{X}^{\text{str}}_\bullet$ is smooth, we have the strictly simplicial ∞-topoi $\mathcal{X}^{\text{str}}_{\bullet,\text{lis-et}}$ and $\mathcal{X}^{\text{str}}_{\bullet,\text{et}}$ (Definition 3.7.3). Explicitly, denoting $\tau =$ lis-et or et, for each $n \in \mathbb{N}$, we define $\mathcal{X}_{n,\tau}$ to be the ∞-topos associated to the ∞-site $\operatorname{LE}(\mathcal{X}_n)$ or $\operatorname{Et}(\mathcal{X}_n)$, and for each morphism $\delta : [m] \to [n]$ we have the corresponding geometric morphism $\delta^{-1}_{\tau} : \mathcal{X}_{n,\tau} \to \mathcal{X}_{m.\tau}$.

By restriction, we have an equivalence $\varepsilon_n : \mathcal{X}^{\text{str}}_{\bullet,\text{lis-et}} \xrightarrow{\sim} \mathcal{X}^{\text{str}}_{\bullet,\text{et}}$ of ∞-topoi for each $n \in \mathbb{N}$. These induce an equivalence

$$\varepsilon : \mathcal{X}^{\text{str}}_{\bullet,\text{lis-et}} \xrightarrow{\sim} \mathcal{X}^{\text{str}}_{\bullet,\text{et}}.$$

Now let $\mathcal{A} \in \mathsf{Shv}_{\mathsf{Com}}(\mathcal{X}_{\text{lis-et}})$ be a flat lisse-étale sheaf of commutative rings. Then the above argument works for the ringed ∞-topos $(\mathcal{X}_{\text{lis-et}}, \mathcal{A})$. Thus we have a strictly simplicial ringed ∞-topos $(\mathcal{X}^{\text{str}}_{\bullet,\text{lis-et}}, \mathcal{A}_\bullet)$ and its restriction $(\mathcal{X}^{\text{str}}_{\bullet,\text{et}}, \mathcal{A}_\bullet)$. We have the equivalence induced by restriction:

$$\varepsilon : \big(\mathcal{X}^{\text{str}}_{\bullet,\text{lis-et}}, \mathcal{A}_\bullet\big) \longrightarrow \big(\mathcal{X}^{\text{str}}_{\bullet,\text{et}}, \mathcal{A}_\bullet\big).$$

We also have the simplicial ringed ∞-topos $(\mathcal{X}_{\bullet,\text{et}}, \mathcal{A}_\bullet)$.

We will analyze objects in $\mathsf{Mod}^{\text{cart}}_{\mathcal{A}}(\mathcal{X}_{\text{lis-et}})$ via these (strictly) simplicial ∞-topoi $\mathcal{X}^{\text{str}}_{\bullet,\text{lis-et}}$, $\mathcal{X}^{\text{str}}_{\bullet,\text{et}}$. and $\mathcal{X}_{\bullet,\text{et}}$. For that, recalling Corollary 5.2.9 which says that $\operatorname{h}\mathsf{Mod}^{\text{cart}}_{\mathcal{A}}(\mathcal{X}_{\text{lis-et}}) \subset \operatorname{h}\mathsf{Mod}_{\mathcal{A}}(\mathcal{X}_{\text{lis-et}})$ is a Serre subcategory, we apply the notations in §3.7.4 to the present situation.

Definition 5.3.2. Let $\mathcal{X}$ and $\mathcal{A}$ be as above.

(1) Let $\tau =$ lis-et or et. An object $\mathcal{M}_\bullet$ of $\mathsf{Mod}_{\mathcal{A}_\bullet}(\mathcal{X}^{\text{str}}_{\bullet,\text{lis-et}})$ is called *cartesian* if $\mathcal{M}_n \in \mathsf{Mod}_{\mathcal{A}_n}(\mathcal{X}_{n,\text{lis-et}})$ is cartesian (Definition 5.2.7) for each $[n] \in \boldsymbol{\Delta}^{\text{str}}$, and if the morphism $\delta^*\mathcal{M}_n \to \mathcal{M}_m$ is an equivalence in $\mathsf{Mod}^{\text{cart}}_{\mathcal{A}_m}(\mathcal{X}_{m,\text{lis-et}})$ for each morphism $\delta : [m] \to [n]$ in $\boldsymbol{\Delta}^{\text{str}}$. We denote by

$$\mathsf{Mod}^{\text{cart}}_{\mathcal{A}_\bullet}(\mathcal{X}_{\bullet,\text{lis-et}}) \subset \mathsf{Mod}_{\mathcal{A}_\bullet}(\mathcal{X}_{\bullet,\text{lis-et}})$$

the full sub-∞-category spanned by cartesian objects.

(2) A *cartesian* object of $\mathsf{Mod}_{\mathcal{A}_\bullet}(\mathcal{X}_{\bullet,\mathrm{et}})$ is defined by the same condition as (1) but replacing $\mathbf{\Delta}^{\mathrm{str}}$ by $\mathbf{\Delta}$. We denote by

$$\mathsf{Mod}^{\mathrm{cart}}_{\mathcal{A}_\bullet}(\mathcal{X}_{\bullet,\mathrm{et}}) \subset \mathsf{Mod}_{\mathcal{A}_\bullet}(\mathcal{X}_{\bullet,\mathrm{et}})$$

the corresponding full sub-∞-category.

We then have the following functors of ∞-categories.

$$\mathsf{Mod}^{\mathrm{cart}}_{\mathcal{A}_\bullet}(\mathcal{X}_{\bullet,\mathrm{et}}) \xrightarrow{\mathrm{Rst}} \mathsf{Mod}^{\mathrm{cart}}_{\mathcal{A}_\bullet}(\mathcal{X}^{\mathrm{str}}_{\bullet,\mathrm{et}}) \xrightarrow{\varepsilon^*} \mathsf{Mod}^{\mathrm{cart}}_{\mathcal{A}_\bullet}(\mathcal{X}^{\mathrm{str}}_{\bullet,\text{lis-et}}).$$

Here Rst denotes the functor induced by the restriction to $\mathbf{\Delta}^{\mathrm{str}} \subset \mathbf{\Delta}$, and ε^* is the one induced by the equivalence ε.

We have the following analogue of [O1, Proposition 4.4].

Proposition 5.3.3. The functors Rst and ε^* are equivalences, and the ∞-categories

$$\mathsf{Mod}^{\mathrm{cart}}_{\mathcal{A}}(\mathcal{X}_{\text{lis-et}}), \quad \mathsf{Mod}^{\mathrm{cart}}_{\mathcal{A}_\bullet}(\mathcal{X}_{\bullet,\mathrm{et}}). \quad \mathsf{Mod}^{\mathrm{cart}}_{\mathcal{A}_\bullet}(\mathcal{X}^{\mathrm{str}}_{\bullet,\mathrm{et}}), \quad \mathsf{Mod}^{\mathrm{cart}}_{\mathcal{A}_\bullet}(\mathcal{X}^{\mathrm{str}}_{\bullet,\text{lis-et}}).$$

are all equivalent.

Proof. We follow the proof of [O1, Proposition 4.4]. Since ε is an equivalence, the induced ε^* is also an equivalence. The fully faithfulness of Rst is obtained by unwinding the definition, so we focus on the essential surjectivity of Rst.

Recall that the 0-th part of $\mathcal{X}_\bullet$ is given by $\mathcal{X}_0 = \coprod_{i\in I} \mathcal{X}_i$, and it is a derived algebraic space. Recalling also that $\mathcal{X}_1 = \mathcal{X}_0 \times_{\mathcal{X}} \mathcal{X}_0$ and $\mathcal{X}_2 = \mathcal{X}_0 \times_{\mathcal{X}} \mathcal{X}_0 \times_{\mathcal{X}} \mathcal{X}_0$, let us denote by $d^1_0, d^1_1 : \mathcal{X}_1 \to \mathcal{X}_0$ and $d^2_0, d^2_1, d^2_2 : \mathcal{X}_2 \to \mathcal{X}_1$ the projections. Compared to the ordinary symbols, we have $d^1_0 = \mathrm{pr}_2$, $d^1_1 = \mathrm{pr}_1$ and $d^2_0 = \mathrm{pr}_{23}$, $d^2_1 = \mathrm{pr}_{13}$, $d^2_2 = \mathrm{pr}_{12}$.

Consider the ∞-category $\mathsf{Des}(\mathcal{X}_0/\mathcal{X})$ whose object is a pair $(\mathcal{G}, \iota)$ of $\mathcal{G} \in \mathsf{Mod}_{\mathcal{A}_0}(\mathcal{X}_{0,\mathrm{et}})$ and an equivalence $\iota : (d^1_1)^*\mathcal{G} \xrightarrow{\sim} (d^1_0)^*\mathcal{G}$ in $\mathsf{Mod}_{\mathcal{A}}(\mathcal{X}_{1,\mathrm{et}})$ such that the two equivalences $(d^2_1)^*(\iota)$ and $(d^0_2)^* \circ (d^2_2)^*$ seen as $(d^1_1 \circ d^2_2)^*\mathcal{G} \to (d^1_0 \circ d^2_0)^*\mathcal{G}$ are equivalent. We have a functor $\mathsf{Mod}^{\mathrm{cart}}_{\mathcal{A}}(\mathcal{X}_{\text{lis-et}}) \to \mathsf{Des}(\mathcal{X}_0/\mathcal{X})$ by sending $\mathcal{M}$ to the pair of $\mathcal{G} := \mathcal{M}_{\mathcal{X}_0}$ (using Notation 5.2.2) and ι defined to be the composition of $(d^1_1)^*\mathcal{M}_{\mathcal{X}_0} \xrightarrow{\sim} \mathcal{M}_{\mathcal{X}_1}$ and the inverse of $(d^1_0)^*\mathcal{M}_{\mathcal{X}_0} \xrightarrow{\sim} \mathcal{M}_{\mathcal{X}_1}$. Then the argument in [O1, Lemma 4.5] works and this functor gives an equivalence.

On the other hand, we have a functor $\mathsf{Mod}^{\mathrm{cart}}_{\mathcal{A}_\bullet}(\mathcal{X}^{\mathrm{str}}_{\bullet,\mathrm{et}}) \to \mathsf{Des}(\mathcal{X}_0/\mathcal{X})$ by sending $\mathcal{M}_\bullet$ to the pair of $\mathcal{G} := \mathcal{M}_0$ and ι defined to be the composition of $(d^1_1)^*\mathcal{M}_0 \xrightarrow{\sim} \mathcal{M}_1$ and the inverse of $(d^1_0)^*\mathcal{M}_0 \xrightarrow{\sim} \mathcal{M}_1$. Note that the last equivalences come from the simplicial structure. Then, by the argument in [O1, Proposition 4.4], this functor is also an equivalence. The same construction gives $\mathsf{Mod}^{\mathrm{cart}}_{\mathcal{A}_\bullet}(\mathcal{X}_{\bullet,\mathrm{et}}) \to \mathsf{Des}(\mathcal{X}_0/\mathcal{X})$. Finally, the composition $\mathsf{Mod}^{\mathrm{cart}}_{\mathcal{A}_\bullet}(\mathcal{X}^{\mathrm{str}}_{\bullet,\mathrm{et}}) \xrightarrow{\sim} \mathsf{Des}(\mathcal{X}_0/\mathcal{X}) \xleftarrow{\sim} \mathsf{Mod}^{\mathrm{cart}}_{\mathcal{A}_\bullet}(\mathcal{X}_{\bullet,\mathrm{et}}) \xrightarrow{\mathrm{Rst}} \mathsf{Mod}^{\mathrm{cart}}_{\mathcal{A}_\bullet}(\mathcal{X}^{\mathrm{str}}_{\bullet,\mathrm{et}})$ is equivalent to the identity functor. Thus Rst is an equivalence,and we have shown that $\mathsf{Mod}^{\mathrm{cart}}_{\mathcal{A}_\bullet}(\mathcal{X}_{\bullet,\mathrm{et}})$. $\mathsf{Mod}^{\mathrm{cart}}_{\mathcal{A}_\bullet}(\mathcal{X}^{\mathrm{str}}_{\bullet,\mathrm{et}})$ and $\mathsf{Mod}^{\mathrm{cart}}_{\mathcal{A}_\bullet}(\mathcal{X}^{\mathrm{str}}_{\bullet,\text{lis-et}})$ are equivalent to $\mathsf{Des}(\mathcal{X}_0/\mathcal{X}) \simeq \mathsf{Mod}^{\mathrm{cart}}_{\mathcal{A}}(\mathcal{X}_{\text{lis-et}})$. □

We next discuss the derived ∞-category. For τ = lis-et or et, we denote by

$$\mathsf{D}_\infty(\mathcal{X}^{\mathrm{str}}_{\bullet,\tau}, \mathcal{A}_\bullet)$$

the derived ∞-category of the Grothendieck abelian category $\mathrm{h}\,\mathsf{Mod}_{\mathcal{A}_\bullet}(\mathcal{X}^{\mathrm{str}}_{\bullet,\tau})$ (Notation 3.7.6). Similarly we define $\mathsf{D}_\infty(\mathcal{X}_{\bullet,\mathrm{et}}, \mathcal{A}_\bullet)$. An object $\mathcal{M}_\bullet$ of $\mathsf{D}_\infty(\mathcal{X}^{\mathrm{str}}_{\bullet,\tau}, \mathcal{A}_\bullet)$ consists of $\mathcal{M}_n \in \mathsf{D}_\infty(\mathcal{X}_{n,\tau}, \mathcal{A}_n)$ for each $n \in \mathbb{N}$ and $\delta^*\mathcal{M}_n \to \mathcal{M}_m$ for each $\delta : [m] \to [n]$ in $\mathbf{\Delta}^{\mathrm{str}}$.

Definition. Let $\mathcal{X}$ and $\mathcal{A}$ be as before.

(1) Let τ = lis-et or et. An object $\mathcal{M}_\bullet$ of $\mathsf{D}_\infty(\mathcal{X}^{\mathrm{str}}_{\bullet,\tau}, \mathcal{A}_\bullet)$ is called *cartesian* if $\mathcal{M}_n$ belongs to $\mathsf{D}^{\mathrm{cart}}_\infty(\mathcal{X}^{\mathrm{str}}_{n,\tau}, \mathcal{A}_n)$ for each $[n] \in \mathbf{\Delta}^{\mathrm{str}}$ and the morphism $\delta^*\mathcal{M}_n \to \mathcal{M}_m$ is an equivalence in $\mathsf{D}_\infty(\mathcal{X}^{\mathrm{str}}_{m,\tau}, \mathcal{A}_m)$ for each morphism $\delta : [m] \to [n]$ in $\mathbf{\Delta}^{\mathrm{str}}$. We denote by

$$\mathsf{D}^{\mathrm{cart}}_\infty(\mathcal{X}^{\mathrm{str}}_{\bullet,\tau}, \mathcal{A}_\bullet) \subset \mathsf{D}_\infty(\mathcal{X}^{\mathrm{str}}_{\bullet,\tau}, \mathcal{A}_\bullet)$$

the full sub-∞-category spanned by cartesian objects.

(2) A *cartesian* object of $\mathsf{D}_\infty(\mathcal{X}_{\bullet,\mathrm{et}}, \mathcal{A}_\bullet)$ is defined by the same condition as (1) with replacing $\mathbf{\Delta}^{\mathrm{str}}$ by $\mathbf{\Delta}$. We denote by

$$\mathsf{D}^{\mathrm{cart}}_\infty(\mathcal{X}_{\bullet,\mathrm{et}}, \mathcal{A}_\bullet) \subset \mathsf{D}_\infty(\mathcal{X}_{\bullet,\mathrm{et}}, \mathcal{A}_\bullet)$$

the corresponding full sub-∞-category.

Let us denote by $\pi : \mathcal{X}^{\mathrm{str}}_{\bullet,\text{lis-et}} \to \mathcal{X}_{\text{lis-et}}$ the functor induced by the strictly simplicial structure, which is indeed a geometric morphisms of ∞-topoi. Recall the equivalence $\varepsilon : \mathcal{X}^{\mathrm{str}}_{\bullet,\text{lis-et}} \xrightarrow{\sim} \mathcal{X}^{\mathrm{str}}_{\bullet,\mathrm{et}}$. We denote by the same symbols the induced functors

$$\pi : (\mathcal{X}^{\mathrm{str}}_{\bullet,\text{lis-et}}, \mathcal{A}_\bullet) \longrightarrow (\mathcal{X}_{\text{lis-et}}, \mathcal{A}), \quad \varepsilon : (\mathcal{X}^{\mathrm{str}}_{\bullet,\text{lis-et}}, \mathcal{A}_\bullet) \longrightarrow (\mathcal{X}^{\mathrm{str}}_{\bullet,\mathrm{et}}, \mathcal{A}_\bullet)$$

of flat ringed ∞-topoi, and denote the corresponding inverse images by

$$\pi^* : \mathsf{D}^{\mathrm{cart}}_\infty(\mathcal{X}_{\text{lis-et}}, \mathcal{A}) \longrightarrow \mathsf{D}^{\mathrm{cart}}_\infty(\mathcal{X}^{\mathrm{str}}_{\bullet,\text{lis-et}}, \mathcal{A}_\bullet), \quad \varepsilon^* : \mathsf{D}^{\mathrm{cart}}_\infty(\mathcal{X}^{\mathrm{str}}_{\bullet,\mathrm{et}}, \mathcal{A}_\bullet) \longrightarrow \mathsf{D}^{\mathrm{cart}}_\infty(\mathcal{X}^{\mathrm{str}}_{\bullet,\text{lis-et}}, \mathcal{A}_\bullet).$$

Let us also define a functor

$$\mathsf{D}^{\mathrm{cart}}_\infty(\mathcal{X}_{\bullet,\mathrm{et}}, \mathcal{A}_\bullet) \xrightarrow{\mathrm{Rst}} \mathsf{D}^{\mathrm{cart}}_\infty(\mathcal{X}^{\mathrm{str}}_{\bullet,\mathrm{et}}, \mathcal{A}_\bullet)$$

by the restriction with respect to $\mathbf{\Delta}^{\mathrm{str}} \subset \mathbf{\Delta}$. Now we can state the main result of this part, which is a derived analogue of [O1, Theorem 4.7].

Theorem 5.3.4. All of the functors π^*, ε^* and Rst are equivalences. In particular, the ∞-categories

$$\mathsf{D}^{\mathrm{cart}}_\infty(\mathcal{X}_{\text{lis-et}}, \mathcal{A}), \quad \mathsf{D}^{\mathrm{cart}}_\infty(\mathcal{X}^{\mathrm{str}}_{\bullet,\text{lis-et}}, \mathcal{A}_\bullet), \quad \mathsf{D}^{\mathrm{cart}}_\infty(\mathcal{X}^{\mathrm{str}}_{\bullet,\mathrm{et}}, \mathcal{A}_\bullet), \quad \mathsf{D}^{\mathrm{cart}}_\infty(\mathcal{X}_{\bullet,\mathrm{et}}, \mathcal{A}_\bullet)$$

are all equivalent, and hence each of them is stable and equipped with a t-structure.

Proof. π^* is an equivalence by Proposition 3.7.8. ε^* is an equivalence since ε is an equivalence. As for Rst, the argument in Proposition 5.3.3 works. ☐

5.4. **Constructible sheaves.** In this subsection Λ denotes a torsion noetherian ring annihilated by an integer invertible in the base commutative ring k.

Recall the notion of a *constructible sheaf* on an ordinary scheme [SGA4, IX §2]:

Definition. A sheaf $\mathcal{F}$ of sets on a scheme X is *constructible* if for any affine Zariski open $U \subset X$ there is a finite decomposition $U = \sqcup_i U_i$ into constructible locally closed subschemes U_i such that $\mathcal{F}|_{U_i}$ is a locally constant sheaf with value in a finite set.

As noted in [LM, Remarque (18.1.2) (1)], the phrase "for any affine Zariski open $U \subset X$" can be replaced by "for each affine scheme belonging to any étale covering of X". In [LM, Chap. 18] the corresponding notion is introduced for the lisse-étale topos of an algebraic stacks. In this subsection we introduce an analogous notion for derived stacks.

We cite from [Lur2, §A.1] the definition of locally constant sheaf on an ∞-topos.

Definition 5.4.1 ([Lur2, Definintion A.1.12])**.** Let T be an ∞-topos and $\mathcal{F} \in \mathsf{T}$.

1. $\mathcal{F}$ is *constant* if it lies in the essential image of a geometric morphism $\pi^* : \mathcal{S} \to \mathsf{T}$ (see Remark 3.1.2 for the notation on a geometric morphism).
2. $\mathcal{F}$ is *locally constant* if there exists a collection $\{U_i\}_{i \in I}$ of objects $U_i \in \mathsf{T}$ satisfying the following conditions.
   - $\{U_i\}_{i \in I}$ covers T, i.e., there is an effective epimorphism $\coprod_{i \in I} U_i \to \mathbf{1}_{\mathsf{T}}$, where $\mathbf{1}_{\mathsf{T}}$ denotes a final object of T (see Definition 1.8.10 for the definition of *effective epimorphism*).
   - The product $\mathcal{F} \times U_i$ is a constant object of the ∞-topos $\mathsf{T}_{/U_i}$ (Fact 1.8.6).

In order to introduce constructible sheaves on a derived stack, we start with those on a derived algebraic space.

Definition 5.4.2. Let $\mathcal{U}$ be a derived algebraic space and $\mathcal{F} \in \mathsf{Shv}_{\mathcal{S}}(\mathcal{U}_{\mathrm{et}}) \simeq \mathcal{U}_{\mathrm{et}}$.

1. $\mathcal{F}$ is *locally constant* if it is locally constant as an object of the ∞-topos $\mathcal{U}_{\mathrm{et}}$ in the sense of Definition 5.4.1.
2. $\mathcal{F}$ is *constructible* if for any étale covering $\{\mathcal{U}_i \to \mathcal{U}\}_{i \in I}$ by derived algebraic spaces $\mathcal{U}_i$, there is a finite decomposition $\mathcal{U}_i \simeq \coprod_j U_{i,j}$ into affine derived schemes $U_{i,j}$ for each i such that
   - The (non-derived) affine scheme $\mathrm{Trc}(U_{i,j})$ is a constructible locally closed subscheme of $\mathrm{Trc}(\mathcal{U}_i)$.
   - The restriction $\mathcal{F}_{U_{i,j}}$ is a locally constant sheaf on $U_{i,j}$ for all j in the sense of (1).

Let us now introduce *constructible sheaves of* Λ*-modules.* As before, for a commutative ring Λ, we denote the constant lisse-étale sheaf valued in Λ by the same symbol.

Definition 5.4.3. Let $\mathcal{U}$ be a derived algebraic space and Λ be a commutative ring. An object $\mathcal{M}$ of $\mathsf{Mod}_\Lambda(\mathcal{U}_{\mathrm{et}})$ is called *constructible* if is is constructible in the sense of Definition 5.4.2 as an object of $\mathsf{Shv}_{\mathcal{S}}(\mathcal{U}_{\mathrm{et}})$.

The following statement is an analogue of [O1, Lemma 9.1]. The proof is the same with loc. cit., and we omit it.

Lemma 5.4.4. Let $\mathcal{X}$ be a geometric derived stack

(1) Let $\mathcal{F} \in \mathsf{Shv}_{\mathcal{S}}(\mathcal{X}_{\text{lis-et}}) \simeq \mathcal{X}_{\text{lis-et}}$ be cartesian. Then the following conditions are equivalent.

  (i) For any $\mathcal{U} \in \mathsf{dAS}_{/\mathcal{X}}$, the sheaf $\mathcal{F}_{\mathcal{U}}$ is a locally constant (resp. constructible) sheaf on $\mathcal{U}_{\text{et}}$.

  (ii) There exists a smooth presentation $\mathcal{X}_\bullet \to \mathcal{X}$ (Definition 5.3.1) such that $\mathcal{F}_{\mathcal{X}_0}$ is locally constant (resp. constructible).

(2) Let Λ be a commutative ring. Then the same equivalences as (1) hold for $\mathcal{M} \in \mathsf{Mod}^{\text{cart}}_{\Lambda}(\mathcal{X}_{\text{lis-et}})$.

Note that the symbol $\mathsf{Mod}^{\text{cart}}_{\Lambda}(\mathcal{X}_{\text{lis-et}})$ in the item (2) makes sense since the constant sheaf Λ is flat (Definition 5.2.4) so that the notion of cartesian sheaves is well-defined.

Now we can state the definition of constructible lisse-étale sheaves on a derived stack.

Definition 5.4.5. Let $\mathcal{X}$ be a geometric derived stack and Λ be a commutative ring. Then a sheaf $\mathcal{M} \in \mathsf{Mod}^{\text{cart}}_{\Lambda}(\mathcal{X}_{\text{lis-et}})$ is *constructible* if it satisfies one of the conditions in Lemma 5.4.4. We denote by

$$\mathsf{Mod}_{\text{c}}(\mathcal{X}_{\text{lis-et}}, \Lambda) \subset \mathsf{Mod}^{\text{cart}}_{\Lambda}(\mathcal{X}_{\text{lis-et}})$$

the full sub-∞-category spanned by constructible sheaves.

Similarly to the case of cartesian sheaves (§5.3), we can describe constructible sheaves by (strictly) simplicial ∞-topoi. Let us take a smooth presentation $\mathcal{X}_\bullet \to \mathcal{X}$, and consider the strictly simplicial ∞-topoi $\mathcal{X}^{\text{str}}_{\bullet,\text{lis-et}}$, $\mathcal{X}^{\text{str}}_{\bullet,\text{et}}$ and the simplicial ∞-topoi $\mathcal{X}_{\bullet,\text{et}}$. These can be ringed by the constant sheaf Λ.

Definition. (1) Let $\tau =$ lis-et or et. An object $\mathcal{M}_\bullet$ of $\mathsf{Mod}^{\text{cart}}_{\Lambda}(\mathcal{X}^{\text{str}}_{\bullet,\tau})$ is *constructible* if $\mathcal{M}_n$ is constructible in the sense of Definition 5.4.5 for each $[n] \in \mathbf{\Delta}^{\text{str}}$. We denote by

$$\mathsf{Mod}_{\text{c}}(\mathcal{X}^{\text{str}}_{\bullet,\tau}, \Lambda) \subset \mathsf{Mod}^{\text{cart}}_{\Lambda}(\mathcal{X}^{\text{str}}_{\bullet,\tau})$$

the full sub-∞-category spanned by constructible objects.

(2) An object of $\mathsf{Mod}^{\text{cart}}_{\Lambda}(\mathcal{X}_{\bullet,\text{et}})$ is defined to be *constructible* in the same way but replacing $\mathbf{\Delta}^{\text{str}}$ by $\mathbf{\Delta}$. The corresponding full sub-∞-category is denoted by

$$\mathsf{Mod}_{\text{c}}(\mathcal{X}_{\bullet,\text{et}}, \Lambda) \subset \mathsf{Mod}^{\text{cart}}_{\Lambda}(\mathcal{X}_{\bullet,\text{et}}).$$

Proposition 5.4.6. The natural restriction functors

$$\mathsf{Mod}_{\text{c}}(\mathcal{X}_{\text{lis-et}}, \Lambda) \longrightarrow \mathsf{Mod}_{\text{c}}(\mathcal{X}^{\text{str}}_{\bullet,\text{lis-et}}, \Lambda) \longrightarrow \mathsf{Mod}_{\text{c}}(\mathcal{X}^{\text{str}}_{\bullet,\text{et}}, \Lambda),$$

and

$$\mathsf{Mod}_{\text{c}}(\mathcal{X}_{\text{lis-et}}, \Lambda) \longrightarrow \mathsf{Mod}_{\text{c}}(\mathcal{X}_{\bullet,\text{et}}, \Lambda) \longrightarrow \mathsf{Mod}_{\text{c}}(\mathcal{X}^{\text{str}}_{\bullet,\text{et}}, \Lambda)$$

are equivalences.

Proof. Let $\mathcal{M}_\bullet \in \mathsf{Mod}^{\text{cart}}_{\Lambda}(\mathcal{X}^{\text{str}}_{\bullet,\tau})$ with $\tau =$ lis-et or et. Recall that by the cartesian condition the morphism $\delta^*\mathcal{M}_m \to \mathcal{M}_n$ is an equivalence for every $\delta : [m] \to [n]$. Thus $\mathcal{M}_\bullet$ is constructible if and only if $\mathcal{M}_0 \in \mathsf{Mod}^{\text{cart}}_{\Lambda}(\mathcal{X}^{\text{str}}_{0,\text{lis-et}})$ is constructible. The same criterion holds for $\mathcal{M}_\bullet \in \mathsf{Mod}^{\text{cart}}_{\Lambda}(\mathcal{X}_{\bullet,\text{et}})$. Now the statement follows from Proposition 5.3.3. □

Let us turn to derived ∞-categories. Recall the stable ∞-category $\mathsf{D}^{\text{cart}}_{\infty}(\mathcal{X}_{\text{lis-et}}, \mathcal{A})$ of cartesian sheaves in Notation 5.2.10. Since the cartesian sheaves form a Serre subcategory (Corollary 5.2.9), the following definition makes sense.

Definition 5.4.7. Let $\mathcal{X}$ be a geometric derived stack, and Λ be a commutative ring. We denote by

$$\mathsf{D}_{\infty,\text{c}}(\mathcal{X}_{\text{lis-et}}, \Lambda) \subset \mathsf{D}_{\infty}(\mathcal{X}_{\text{lis-et}}, \Lambda)$$

the full sub-∞-category spanned by those objects $\mathcal{M}$ whose homotopy sheaf $\pi_j\mathcal{M} \in \mathsf{Mod}_{\Lambda}(\mathcal{X}_{\text{lis-et}})$ is constructible (Definition 5.4.5) for any $j \in \mathbb{Z}$. We also denote by

$$\mathsf{D}^{*}_{\infty,\text{c}}(\mathcal{X}_{\text{lis-et}}, \Lambda) \subset \mathsf{D}_{\infty,\text{c}}(\mathcal{X}_{\text{lis-et}}, \Lambda) \quad (* \in \{+, -, b, [m,n]\})$$

the full sub-∞-category spanned by constructible objects which are right bounded, left bounded, bounded and bounded in the range $[m, n]$.

As in the previous §5.3, we can describe $\mathsf{D}_{\infty,\text{c}}(\mathcal{X}_{\text{lis-et}}, \Lambda)$ by (strictly) simplicial ∞-topoi.

Definition. (1) Let $\tau =$ lis-et or et. An object $\mathcal{M}_\bullet$ of $\mathsf{D}^{\text{cart}}_{\infty}(\mathcal{X}^{\text{str}}_{\bullet,\tau}, \Lambda)$ is *constructible* if $\mathcal{M}_n$ is constructible in the sense of Definition 5.4.7 for each $[n] \in \mathbf{\Delta}^{\text{str}}$. We denote by

$$\mathsf{D}_{\infty,\text{c}}(\mathcal{X}^{\text{str}}_{\bullet,\tau}, \Lambda) \subset \mathsf{D}_{\infty,\text{c}}(\mathcal{X}^{\text{str}}_{\bullet,\tau}, \Lambda)$$

the full sub-∞-category spanned by constructible objects.

(2) An object of $\mathsf{D}^{\mathrm{cart}}_{\infty}(\mathcal{X}_{\bullet,\mathrm{et}}, \Lambda)$ is defined to be *constructible* in the same way but replacing $\mathbf{\Delta}^{\mathrm{str}}$ by $\mathbf{\Delta}$. The corresponding full sub-∞-category is denoted by

$$\mathsf{D}_{\infty,\mathrm{c}}(\mathcal{X}_{\bullet,\mathrm{et}}, \Lambda) \subset \mathsf{D}_{\infty,\mathrm{c}}(\mathcal{X}_{\bullet,\mathrm{et}}, \Lambda).$$

As in §5.3, we denote by

$$\pi : \mathcal{X}^{\mathrm{str}}_{\bullet,\text{lis-et}} \longrightarrow \mathcal{X}_{\text{lis-et}}, \quad \varepsilon : \mathcal{X}^{\mathrm{str}}_{\bullet,\text{lis-et}} \longrightarrow \mathcal{X}^{\mathrm{str}}_{\bullet,\mathrm{et}}$$

the natural geometric morphism of ∞-topoi. We denote the induce functors of derived ∞-categories by

$$\pi^* : \mathsf{D}_{\infty,\mathrm{c}}(\mathcal{X}_{\text{lis-et}}, \Lambda) \longrightarrow \mathsf{D}_{\infty,\mathrm{c}}(\mathcal{X}^{\mathrm{str}}_{\bullet,\text{lis-et}}, \Lambda), \quad \varepsilon^* : \mathsf{D}_{\infty,\mathrm{c}}(\mathcal{X}^{\mathrm{str}}_{\bullet,\mathrm{et}}, \Lambda) \longrightarrow \mathsf{D}_{\infty,\mathrm{c}}(\mathcal{X}^{\mathrm{str}}_{\bullet,\text{lis-et}}, \Lambda).$$

Theorem 5.4.8. Both π^* and ε^* are equivalences. In particular, $\mathsf{D}^*_{\infty,\mathrm{c}}(\mathcal{X}_{\text{lis-et}}, \Lambda)$ is stable and equipped with a t-structure.

Proof. This is a direct consequence of Theorem 5.3.4 and definitions. □

5.5. **Dualizing objects for derived stacks.** In this subsection we introduce dualizing complexes for the derived category of constructible lisse-étale sheaves on derived stacks. which will be used in the construction of derived functors (§6). The strategy of defining dualizing complexes follows that in [LO1, 3.3–3.5]: gluing dualizing complexes over derived algebraic spaces (§4.6) to obtain the global one.

5.5.1. *Localized ∞-sites and localized ∞-topoi.* As a preliminary, let us introduce *localized ∞-sites* explain the relation to the localized ∞-topoi.

Let (C, τ) be an ∞-site, and $X \in \mathsf{C}$ be an object. Then we can attach to the over-∞-category $\mathsf{C}_{/X}$ a new Grothendieck topology τ_X by setting the covering sieve to be

$$\mathrm{Cov}_{\tau_X}(\varphi) := \{\varphi^* \mathsf{C}^{(0)}_{/X} \mid \mathsf{C}^{(0)}_{/X} \in \mathrm{Cov}_{\tau}(X)\}$$

for each $\varphi \in \mathsf{C}_{/X}$. Let us name the obtained ∞-site.

Definition 5.5.1. The ∞-site $(\mathsf{C}_{/X}, \tau_X)$ is called the *localized ∞-site* on X.

In [TVe1, Definition 3.3.1] the corresponding S-site is called the comma S-site.

The following statement is checked directly by definition so we omit the proof.

Lemma 5.5.2. Let (C, τ) be an ∞-site and $f : X \to Y$ be a morphism in C. Then composition with f induces a continuous functor $(\mathsf{C}_{/X}, \tau_X) \to (\mathsf{C}_{/Y}, \tau_Y)$ of ∞-sites (Definition 4.2.1).

Recall that for an ∞-topos T and $U \in \mathsf{T}$, we have the localized ∞-topos $\mathsf{T}|_U$ of T at U (Fact 1.8.6). Thus, given an ∞-site (C, τ) and an object $X \in \mathsf{C}$, we can consider the localized ∞-topos $\mathsf{Sh}(\mathsf{C}, \tau)|_{j(X)}$. Here $j : \mathsf{C} \to \mathsf{Shv}(\mathsf{C}, \tau)$ is the ∞-categorical Yoneda embedding (Definition 1.5.2). On the other hand, we have the localized ∞-site $(\mathsf{C}_{/X}, \tau_X)$ and the associated ∞-topos $\mathsf{Sh}(\mathsf{C}_{/X}, \tau_X)$.

Lemma 5.5.3. For an ∞-site (C, τ) and $X \in \mathsf{C}$, we have an equivalence $\mathsf{Sh}(\mathsf{C}_{/X}, \tau_X) \simeq \mathsf{Sh}(\mathsf{C}, \tau)|_{j(X)}$.

Proof. The forgetful functor $\mathsf{C}_{/X} \to \mathsf{C}$ induces a continuous functor $(\mathsf{C}_{/X}, \tau_X) \to \mathsf{Sh}(\mathsf{C}, \tau)$ of ∞-sites. Then by Proposition 4.2.2, we have a geometric morphism $\iota^* : \mathsf{Sh}(\mathsf{C}_{/X}, \tau_X) \to \mathsf{Sh}(\mathsf{C}, \tau)$ of ∞-topoi (recall Notation 3.1.2). By definition of $(\mathsf{C}_{/X}, \tau_X)$, the geometric morphism ι^* factors through $\mathsf{Sh}(\mathsf{C}, \tau)|_{j(X)}$, and the factored geometric morphism $\mathsf{Sh}(\mathsf{C}_{/X}, \tau_X) \to \mathsf{Sh}(\mathsf{C}, \tau)|_{j(X)}$ gives an equivalence. □

Recall that a continuous functor of ∞-sites induces a geometric morphism of the associated ∞-topoi (Proposition 4.2.2). Thus the continuous functor $(\mathsf{C}_{/X}, \tau_X) \to (\mathsf{C}_{/Y}, \tau_Y)$ in Lemma 5.5.2 induces a geometric morphism $\mathsf{Sh}(\mathsf{C}_{/X}, \tau_X) \to \mathsf{Sh}(\mathsf{C}_{/Y}, \tau_Y)$. Using the equivalence of Lemma 5.5.3, we denote it by $\alpha : \mathsf{Sh}(\mathsf{C}, \tau)|_{j(X)} \to \mathsf{Sh}(\mathsf{C}, \tau)|_{j(Y)}$.

On the other hand, for any morphism $U \to V$ in a ∞-category B, we have a functor $\mathsf{B}_{/U} \to \mathsf{B}_{/V}$ (Corollary B.3.2) Thus we have a functor $\beta : \mathsf{Sh}(\mathsf{C}, \tau)|_{j(X)} \to \mathsf{Sh}(\mathsf{C}, \tau)|_{j(Y)}$. Then we have

Lemma 5.5.4. The functors α and β are equivalent.

5.5.2. *Gluing étale dualizing objects.* We impose Assumption 4.6.1 on the base ring k and the commutative ring Λ. Namely,

- The base ring k is a noetherian ring and has a dualizing complex Ω_k.
- The commutative ring Λ is a torsion noetherian ring annihilated by an integer invertible in k.

Let $\mathcal{X}$ be a geometric derived stack locally of finite presentation (Definition 2.2.19). Let $A : \mathcal{U} \to \mathcal{X}$ be an object of $\mathsf{dAS}_{\mathcal{X}}^{\mathrm{lis}}$, the underlying ∞-category of the ∞-site $\mathrm{LE}(\mathcal{X})$. Then the derived algebraic space $\mathcal{U}$ satisfies Assumption 4.6.1:

- The derived algebraic space $\mathcal{W}$ is separated, quasi-compact and locally of finite presentation.

Thus we can apply the argument in §4.6 to the composition $\alpha := (\mathcal{U} \xrightarrow{A} \mathcal{X} \to \mathrm{dSpec}\, k)$, and have the dualizing object $\Omega_\alpha \in \mathsf{D}_\infty(\mathcal{U}_{\mathrm{et}}, \Lambda)$ (Notation 4.6.5). Recalling also Notation 4.6.6 on the shift and the Tate twist, we define

$$K_A := \Omega_\alpha \langle -d_A \rangle \in \mathsf{D}_{\infty,\mathrm{c}}(\mathcal{U}_{\mathrm{et}}, \Lambda),$$

where d_A is the relative dimension of the smooth morphism A (Definition 2.2.23). Note that d_A is locally constant, and that K_A is of finite injective dimension by Lemma 4.6.4.

Now the proof of [LO1, 3.2.1 Lemma] works with the help of Lemma 4.6.7, and we have

Lemma 5.5.5. Given a triangle

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{\ \sigma\ } & \mathcal{U} \\ & {\scriptstyle B}\searrow \quad \swarrow{\scriptstyle A} & \\ & \mathcal{X} & \end{array}$$

in $\mathsf{dAS}_{\mathcal{X}}^{\mathrm{lis}}$, we have an equivalence $\sigma^* K_A \simeq K_B$.

We now want to construct a dualizing object for the lisse-étale by gluing étale dualizing data using Lemma 5.5.5. Let $A : \mathcal{U} \to \mathcal{X}$ be an object of $\mathrm{LE}(\mathcal{X})$. We denote by $\mathrm{LE}(\mathcal{X})|_{\mathcal{U}}$ the localized ∞-site (Definition 5.5.1). Then the inclusion $\mathrm{Et}(\mathcal{U}) \hookrightarrow \mathrm{LE}(\mathcal{X})|_{\mathcal{U}}$ of ∞-sites is a continuous functor, and we can apply Proposition 4.2.2 to it. Thus we have

Lemma 5.5.6. The inclusion $\mathrm{Et}(\mathcal{U}) \hookrightarrow \mathrm{LE}(\mathcal{X})|_{\mathcal{U}}$ induces a geometric morphism

$$\varepsilon_{\mathcal{U}} : \mathcal{X}_{\text{lis-et}}|_{\mathcal{U}} \longrightarrow \mathcal{U}_{\mathrm{et}}$$

of ∞-topoi. We denote the corresponding adjunction by $\varepsilon_{\mathcal{U}}^{-1} : \mathcal{U}_{\mathrm{et}} \rightleftarrows \mathcal{X}_{\text{lis-et}}|_{\mathcal{U}} : \varepsilon_*^{\mathcal{U}}$.

We can describe these functors more explicitly. Note that giving a sheaf $\mathcal{G} \in \mathcal{X}_{\text{lis-et}}|_{\mathcal{U}}$ is equivalent to giving sheaves $\mathcal{G}_V \in V_{\mathrm{et}}$ for each affine derived scheme V over $\mathcal{U}$ such that the composite $V \to \mathcal{U} \xrightarrow{A} \mathcal{X}$ is smooth, which satisfy the gluing condition. Then, for a given $\mathcal{F} \in \mathcal{U}_{\mathrm{et}}$, the sheaf $\mathcal{G} = \varepsilon_{\mathcal{U}}^{-1}\mathcal{F} \in \mathcal{X}_{\text{lis-et}}|_{\mathcal{U}}$ corresponds to $\mathcal{G}_V = \pi_V^{-1}\mathcal{F}$ for each $(\pi_V : V \to \mathcal{U}) \in \mathrm{LE}(\mathcal{X})|_{\mathcal{U}}$. On the other hand, for a sheaf $\mathcal{G} \in \mathcal{X}_{\text{lis-et}}|_{\mathcal{U}}$ corresponding to $\{\mathcal{G}_V\}_{V \to \mathcal{U}}$, the sheaf $\varepsilon_*^{\mathcal{U}}\mathcal{G} \in \mathcal{U}_{\mathrm{et}}$ is given by $\varepsilon_*^{\mathcal{U}}\mathcal{G} = \mathcal{G}_{\mathcal{U}}$. In particular, the functor $\varepsilon_*^{\mathcal{U}}$ is left and right exact.

Given a morphism $f : \mathcal{U} \to \mathcal{V}$ in the underlying ∞-category $\mathsf{dAS}_{\mathcal{X}}^{\mathrm{lis}}$ of the ∞-site $\mathrm{LE}(\mathcal{X})$, we have a square

$$\begin{array}{ccc} \mathcal{X}_{\text{lis-et}}|_{\mathcal{U}} & \xrightarrow{\ \varepsilon_{\mathcal{U}}\ } & \mathcal{U}_{\mathrm{et}} \\ {\scriptstyle f}\downarrow & & \downarrow \\ \mathcal{X}_{\text{lis-et}}|_{\mathcal{V}} & \xrightarrow[\ \varepsilon_{\mathcal{U}}\ ]{} & \mathcal{V}_{\mathrm{et}} \end{array} \tag{5.5.1}$$

in the ∞-category RTop of ∞-topoi and geometric morphisms. Here $f : \mathcal{X}_{\text{lis-et}}|_{\mathcal{U}} \to \mathcal{X}_{\text{lis-et}}|_{\mathcal{V}}$ denotes the functor induced by f (Corollary 5.5.2, Lemma 5.5.4).

Recalling that we are given $A : \mathcal{U} \to \mathcal{X}$, let us now define

$$\kappa_A := \varepsilon_{\mathcal{U}}^*(K_A) \in \mathsf{D}_\infty(\mathcal{X}_{\text{lis-et}}|_{\mathcal{U}}, \Lambda),$$

where we denote by $\varepsilon_{\mathcal{U}}^* : \mathsf{D}_\infty(\mathcal{U}_{\mathrm{et}}, \Lambda) \rightleftarrows \mathsf{D}_\infty(\mathcal{X}_{\text{lis-et}}|_{\mathcal{U}}, \Lambda) : f_*\varepsilon^{\mathcal{U}}$ the geometric morphism of the derived ∞-categories induced by $\varepsilon_{\mathcal{U}}$. Then, given a triangle as in Lemma 5.5.5, we have $f^*(\kappa_B) \simeq \kappa_A$ since (5.5.1) is a square. Recall Notation 3.5.5 of $\mathscr{E}xt$. We denote by $\mathscr{E}xt^i_{\mathcal{U}_{\mathrm{et}}}$ the functor $\mathscr{E}xt_\Lambda$ on $\mathsf{D}_\infty(\mathcal{U}_{\mathrm{et}})$, and by $\mathscr{E}xt^i_{\mathcal{X}_{\text{lis-et}}|_{\mathcal{U}}}$ that on $\mathsf{D}_\infty(\mathcal{X}_{\text{lis-et}}|_{\mathcal{U}})$. Then we have

Lemma 5.5.7. (1) For any $\mathcal{M}, \mathcal{N} \in \mathsf{D}_\infty(\mathcal{U}_{\mathrm{et}})$ and $i \in \mathbb{Z}$, we denote by $\mathscr{E}xt^i(\varepsilon_{\mathcal{U}}^*\mathcal{M}, \varepsilon_{\mathcal{U}}^*\mathcal{N})_{\mathcal{U}} \in \mathsf{Mod}_\Lambda(\mathcal{U}_{\mathrm{et}})$ the restriction of $\mathscr{E}xt^i_{\mathcal{X}_{\text{lis-et}}|_{\mathcal{U}}}(\varepsilon_{\mathcal{U}}^*\mathcal{M}, \varepsilon_{\mathcal{U}}^*\mathcal{N})$ to $\mathcal{U}_{\mathrm{et}}$. Then we have an equivalence

$$\mathscr{E}xt^i(\varepsilon_{\mathcal{U}}^*\mathcal{M}, \varepsilon_{\mathcal{U}}^*\mathcal{N})_{\mathcal{U}} \simeq \mathscr{E}xt^i_{\mathcal{U}_{\mathrm{et}}}(\mathcal{M}, \mathcal{N}).$$

(2) For any $A : \mathcal{U} \to \mathcal{X}$ in $\mathsf{dAS}_{\mathcal{X}}^{\mathrm{lis}}$ we have

$$\mathscr{H}om_\Lambda(\kappa_A, \kappa_A) = \Lambda$$

in $\mathsf{D}_\infty(\mathcal{X}_{\text{lis-et}}|_{\mathcal{U}}, \Lambda)$. In particular, we have $\mathscr{E}xt^i_\Lambda(\kappa_A, \kappa_A) = 0$ in $\mathsf{Mod}_\Lambda(\mathcal{X}_{\text{lis-et}}|_{\mathcal{U}})$ for any $i \in \mathbb{Z}_{<0}$.

We omit the detail of the proof since (1) can be shown by the same argument of [LO1, 3.4.1. Lemma] and (2) is a corollary of (1).

Now choose a smooth presentation $p : \mathcal{X}_\bullet \to \mathcal{X}$. Then we have an object $\kappa_p \in \mathsf{D}_\infty(\mathcal{X}_{\text{lis-et}}|_{\mathcal{X}_0}, \Lambda)$ together with the descent data to $\mathcal{X}_{\text{lis-et}}$. Thus the gluing lemma (Fact 3.7.9) can be applied, and we have an object $\Omega_{\mathcal{X}}(p) \in \mathsf{D}_\infty(\mathcal{X}_{\text{lis-et}}, \Lambda)$. Since κ_A is of finite injective dimension for any $A : \mathcal{U} \to \mathcal{X}$ in $\mathsf{dAS}_{\mathcal{X}}^{\mathrm{lis}}$, the restriction $\kappa_p|_{\mathcal{U}} \simeq \kappa_A$ is bounded in both direction.

Notation 5.5.8. We denote by

$$\mathsf{D}^{(*)}_{\infty,\mathrm{c}}(\mathcal{X}_{\text{lis-et}}, \Lambda) \subset \mathsf{D}_{\infty,\mathrm{c}}(\mathcal{X}_{\text{lis-et}}, \Lambda) \quad (* \in \{+, -, b\})$$

the full sub-∞-category spanned by those objects $\mathcal{M}$ such that the restriction $\mathcal{M}|_{\mathcal{U}}$ is in $\mathsf{D}^*_{\infty,\mathrm{c}}(\mathcal{X}_{\text{lis-et}}, \Lambda)$ for any quasi-compact open immersion $\mathcal{U} \hookrightarrow \mathcal{X}$ (Definition 2.2.12, 2.2.24).

By the above argument, we have

Lemma. There exists $\Omega_{\mathcal{X}}(p) \in \mathsf{D}_\infty(\mathcal{X}_{\text{lis-et}}, \Lambda)$ inducing κ_A for any $A \in \mathrm{LE}(\mathcal{X})|_{\mathcal{X}_0}$. Moreover it is unique up to contractible ambiguity.

We can show that independence of the presentation p, for example, by constructing a new presentation from given two presentations. Thus the following definition makes sense.

Definition 5.5.9. The *dualizing object*

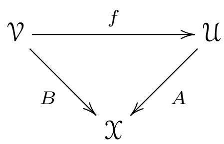
$$\Omega_{\mathcal{X}} \in \mathsf{D}_\infty(\mathcal{X}_{\text{lis-et}}, \Lambda)$$

of $\mathcal{X}$ is defined to be $\Omega_{\mathcal{X}}(p)$ with p a smooth presentation of $\mathcal{X}$. It is well-defined up to contractible ambiguity, and is characterized by $\Omega_{\mathcal{X}}|_{\mathcal{U}} = \varepsilon_{\mathcal{U}}^* K_A$ for $(A : \mathcal{U} \to \mathcal{X}) \in \mathsf{dAS}_{\mathcal{X}}^{\mathrm{lis}}$.

5.5.3. *Biduality.* We impose the same conditions on k and Λ as in the previous §5.5.2 (see also Assumption 4.6.1). Let $\mathcal{X}$ be a geometric derived stack which is locally of finite presentation.

For $(A : \mathcal{U} \to \mathcal{X}) \in \mathsf{dAS}_{\mathcal{X}}^{\mathrm{lis}}$, we denote by $K_A \in \mathsf{D}_\infty(\mathcal{U}_{\mathrm{et}}, \Lambda)$ the dualizing object for the derived algebraic space $\mathcal{U}$. Also, for $\mathcal{M} \in \mathsf{D}_\infty(\mathcal{X}_{\text{lis-et}}, \Lambda)$, we use Notation 5.2.2 to denote by $\mathcal{M}_{\mathcal{U}} \in \mathsf{D}_\infty(\mathcal{U}_{\mathrm{et}}, \Lambda)$ the restriction of $\mathcal{M}$. Then by the argument in [LO1, 3.5] we have

Fact 5.5.10 ([LO1, 3.5.2. Lemma])**.** Let



be a commutative triangle in $\mathsf{dAS}_{\mathcal{X}}^{\mathrm{lis}}$, and let $\mathcal{M} \in \mathsf{D}_\infty(\mathcal{X}_{\text{lis-et}}, \Lambda)$.

(1) In $\mathsf{D}_\infty(\mathcal{V}_{\mathrm{et}}, \Lambda)$ we have $f^*\mathscr{H}om_\Lambda(\mathcal{M}_{\mathcal{U}}, K) = \mathscr{H}om_\Lambda(f^*\mathcal{M}_{\mathcal{U}}, f^*K_A) = \mathscr{H}om_\Lambda(f^*\mathcal{M}_{\mathcal{U}}, K_B)$.
(2) For any $\mathcal{M} \in \mathsf{D}_\infty(\mathcal{X}_{\text{lis-et}}, \Lambda)$, we have $\mathscr{H}om_\Lambda(\mathcal{M}_{\mathcal{U}}, K_A) \in \mathsf{D}_{\infty,\mathrm{c}}(\mathcal{U}_{\mathrm{et}}, \Lambda)$.

For $\mathcal{U} \in \mathsf{dAS}_{\mathcal{X}}^{\mathrm{lis}}$, we denote by $\varepsilon : \mathcal{X}_{\text{lis-et}}|_{\mathcal{U}} \to \mathcal{U}_{\mathrm{et}}$ the geometric morphism in Lemma 5.5.6. Let also $\Omega_{\mathcal{X}}$ be the dualizing object for the derived stack $\mathcal{X}$ (Definition 5.5.9).

Fact 5.5.11 ([LO1, 3.5.3. Lemma])**.** Let $(A : \mathcal{U} \to \mathcal{X}) \in \mathsf{dAS}_{\mathcal{X}}^{\mathrm{lis}}$ and $\mathcal{M} \in \mathsf{D}_{\infty,\mathrm{c}}(\mathcal{X}_{\text{lis-et}})$. Then we have an equivalence $\varepsilon^*\mathscr{H}om_\Lambda(\mathcal{M}_{\mathcal{U}}, K_A) \simeq \mathscr{H}om_\Lambda(\mathcal{M}, \Omega_{\mathcal{X}})_{\mathcal{U}}$.

By Fact 5.5.10 and 5.5.11 we have

Lemma. For $\mathcal{M} \in \mathsf{D}_{\infty,\mathrm{c}}(\mathcal{X}_{\text{lis-et}}, \Lambda)$, we have $\mathscr{H}om_\Lambda(\mathcal{M}, \Omega_{\mathcal{X}}) \in \mathsf{D}_{\infty,\mathrm{c}}(\mathcal{X}_{\text{lis-et}}, \Lambda)$.

Now we can introduce

Notation 5.5.12. The *dualizing functor* is defined to be

$$\mathrm{D}_{\mathcal{X}} := \mathscr{H}om(-, \Omega_{\mathcal{X}}) : \mathsf{D}_{\infty,\mathrm{c}}(\mathcal{X}_{\text{lis-et}}, \Lambda) \longrightarrow \mathsf{D}_{\infty,\mathrm{c}}(\mathcal{X}_{\text{lis-et}}, \Lambda)^{\mathrm{op}}.$$

By the standard argument using Fact 5.5.10 and 5.5.11, we have

Proposition 5.5.13 ([LO1, 3.5.8, 3.5.9. Proposition]). (1) The natural morphism $\mathrm{id} \to \mathrm{D}_{\mathcal{X}} \circ \mathrm{D}_{\mathcal{X}}$ induced by the biduality morphism $\mathcal{M} \to \mathscr{H}om_{\Lambda}(\mathscr{H}om_{\Lambda}(\mathcal{M}, \mathcal{N}), \mathcal{N})$ is an equivalence.

(2) For any $\mathcal{M}, \mathcal{N} \in \mathsf{Mod}_{\mathrm{c}}^{\mathrm{stab}}(\mathcal{X}_{\text{lis-et}}, \Lambda)$ we have a canonical equivalence

$$\mathscr{H}om_{\Lambda}(\mathcal{M}, \mathcal{N}) \simeq \mathscr{H}om_{\Lambda}(\mathrm{D}(\mathcal{N}), \mathrm{D}(\mathcal{M})).$$

6. Derived functors of constructible sheaves with finite coefficients

In this section we introduce ∞-theoretic analogue of the Grothendieck's six operations on the derived categories of constructible sheaves with finite coefficients. Let k be the base commutative ring, and Λ be a Gorenstein local ring of dimension 0 and characteristic ℓ. We assume that ℓ is invertible in k. All the derived stacks appearing in this section are defined over k.

6.1. Derived category of constructible sheaves.

Let us begin with the recollection of the notations for derived ∞-categories of constructible lisse-étale sheaves on derived stacks. Let $\mathcal{X}$ be a geometric derived stack over k which is locally of finite presentation. Then we have the full sub-∞-categories

$$\mathsf{D}^{*}_{\infty,\mathrm{c}}(\mathcal{X}_{\text{lis-et}}, \Lambda) \subset \mathsf{D}^{\mathrm{cart},*}_{\infty}(\mathcal{X}_{\text{lis-et}}, \Lambda) \subset \mathsf{D}^{*}_{\infty}(\mathcal{X}_{\text{lis-et}}, \Lambda) \quad (* \in \{\emptyset, +, -, b\})$$

spanned by constructible (resp. cartesian) objects in the derived ∞-categories of lisse-étale Λ-modules on $\mathcal{X}$ with prescribed bound conditions. These are stable ∞-categories equipped with t-structure by Theorem 5.3.4 and 5.4.8.

We also denote

$$\mathrm{D}^{*}(\mathcal{X}_{\text{lis-et}}, \Lambda) := \mathrm{h}\,\mathsf{D}^{*}_{\infty}(\mathcal{X}_{\text{lis-et}}, \Lambda), \quad \mathrm{D}^{*}_{\mathrm{c}}(\mathcal{X}_{\text{lis-et}}, \Lambda) := \mathrm{h}\,\mathsf{D}^{*}_{\infty,\mathrm{c}}(\mathcal{X}_{\text{lis-et}}, \Lambda).$$

These are triangulated categories by Fact D.1.5, and the obvious embedding $\mathrm{D}^{*}_{\mathrm{c}}(\mathcal{X}, \Lambda) \hookrightarrow \mathrm{D}^{*}(\mathcal{X}, \Lambda)$ is a triangulated functor.

Remark. We have an obvious relation between the derived category for a derived stack and that for an ordinary algebraic stack. Let X be an algebraic stack over k, and $\iota(X)$ be the associated derived stack (Definition 2.2.29). Then by the construction we find that the category $\mathrm{D}^{*}(\iota(X)_{\text{lis-et}}, \Lambda)$ (resp. $\mathrm{D}^{*}_{\mathrm{c}}(\iota(X)_{\text{lis-et}}, \Lambda)$) is equivalent to the derived category of complexes of Λ-modules (resp. the derived category of complexes of Λ-modules with constructible cohomology sheaves).

Let us also recall Notation 5.5.8, where we denoted by

$$\mathsf{D}^{(*)}_{\infty,\mathrm{c}}(\mathcal{X}_{\text{lis-et}}, \Lambda) \subset \mathsf{D}_{\infty,\mathrm{c}}(\mathcal{X}_{\text{lis-et}}, \Lambda) \quad (* \in \{+, -, b\})$$

the full sub-∞-category consisting of objects $\mathcal{M}$ such that for any quasi-compact open immersion $\mathcal{U} \hookrightarrow \mathcal{X}$ the restriction $\mathcal{M}|_{\mathcal{U}}$ belongs to $\mathsf{D}_{\infty,\mathrm{c}}{}^{*}(\mathcal{U}_{\text{lis-et}}, \Lambda)$. The homotopy categories will be denoted by $\mathrm{D}^{(*)}_{\mathrm{c}}(\mathcal{X}_{\text{lis-et}}, \Lambda) := \mathrm{h}\,\mathsf{D}^{(*)}_{\infty,\mathrm{c}}(\mathcal{X}_{\text{lis-et}}, \Lambda)$.

Hereafter we often suppress the symbol Λ to denote $\mathsf{D}_{\infty,\mathrm{c}}(\mathcal{X}_{\text{lis-et}}) := \mathsf{D}_{\infty,\mathrm{c}}(\mathcal{X}_{\text{lis-et}}, \Lambda)$ and so on.

6.2. Derived direct image functor.

Let us recall the derived direct image functor for ordinary algebraic stacks [LO1, 4.1]. Let $f : X \to Y$ be a morphism of finite type between algebraic stacks locally of finite type over k (actually we can relax the condition on the base scheme). We denote by $\mathrm{D}(X)$ the derived category of complexes of Λ-modules on the algebraic stack X. Then the derived direct image functor $\mathrm{R}f_{*} : \mathrm{D}(X) \to \mathrm{D}(Y)$ does not map the subcategory $\mathrm{D}_{\mathrm{c}}(X)$ of complexes with constructible cohomology sheaves to $\mathrm{D}_{\mathrm{c}}(Y)$. However, we have $\mathrm{R}f_{*} : \mathrm{D}^{(+)}_{\mathrm{c}}(X) \to \mathrm{D}^{(+)}_{\mathrm{c}}(Y)$, where $\mathrm{D}^{(+)}_{\mathrm{c}}(X) \subset \mathrm{D}(X)$ is the full subcategory defined similarly as Notation 5.5.8.

Now we consider the case of derived stacks. Recall Proposition 3.5.4 which gives derived functors between derived ∞-categories for general ringed ∞-topoi. We then have the first half of the next proposition.

Proposition 6.2.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of geometric derived stacks. Then we have the direct image functor

$$f_{*} : \mathsf{D}_{\infty}(\mathcal{X}_{\text{lis-et}}) \longrightarrow \mathsf{D}_{\infty}(\mathcal{Y}_{\text{lis-et}}), \quad (f_{*}\mathcal{F})(U) = \mathcal{F}(U \times_{\mathcal{Y}} \mathcal{X})$$

which is a t-exact functor of stable ∞-categories equipped with t-structures. If moreover f is quasi-compact (Definition 2.2.12), then f_{*} restricts to a functor

$$f_{*} : \mathsf{D}^{(+)}_{\infty,\mathrm{c}}(\mathcal{X}_{\text{lis-et}}) \longrightarrow \mathsf{D}^{(+)}_{\infty,\mathrm{c}}(\mathcal{Y}_{\text{lis-et}}).$$

Recall Notation 5.2.2 on the restriction of lisse-étale sheaves. We need the following analogue of [O1, Proposition 9.8]. The proof is almost the same, and we omit it.

Lemma. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of geometric derived stacks, $\mathcal{M} \in \mathsf{Mod}_\Lambda(\mathcal{X}_{\text{lis-et}})$, and $U \in \mathsf{dAS}^{\text{lis}}_{\mathcal{Y}}$.

(1) Assume f is representable. Then we have an equivalence
$$(f_*\mathcal{F})_{\mathcal{U}} \simeq f_{\mathcal{U}_{\text{et}},*}\mathcal{M}_{\mathcal{X}\times_{\mathcal{Y}}\mathcal{U}}$$
in $\mathsf{D}_\infty(\mathcal{U}_{\text{et}})$. Here $f_* : \mathsf{D}_\infty(\mathcal{X}_{\text{lis-et}}) \to \mathsf{D}_\infty(\mathcal{Y}_{\text{lis-et}})$ denotes the derived direct image functor, and $f_{\mathcal{U}_{\text{et}}} : (\mathcal{X}\times_{\mathcal{Y}}\mathcal{U})_{\text{et}} \to \mathcal{U}_{\text{et}}$ is the geometric morphism of ∞-topoi induced by the projection $\mathcal{X}\times_{\mathcal{Y}}\mathcal{U} \to \mathcal{U}$.

(2) Let $\mathcal{X}_\bullet \to \mathcal{X}$ be a smooth presentation, and $f_{\mathcal{U},n} : \mathcal{X}_n \times_{\mathcal{Y}} \mathcal{U} \to \mathcal{U}$ be the morphism induced by f. Assume $\mathcal{M}$ is cartesian. Then there is a spectral sequence
$$E_1^{pq} = R^q(f_{\mathcal{U},n})_*\mathcal{M}_{\mathcal{X}_p\times_{\mathcal{Y}}\mathcal{U}} \Longrightarrow (R^{p+q}f_*\mathcal{M})_{\mathcal{U}}.$$

Proof of Proposition 6.2.1. It is enough to show the second half. By Lemma 5.2.8 of the criterion of cartesian property and by Theorem 5.3.4 of the equivalence $\mathsf{D}_\infty(\mathcal{Y}_{\text{lis-et}}) \simeq \mathsf{D}_\infty(\mathcal{Y}_{\bullet,\text{et}})$, we can take a smooth presentation $\mathcal{Y}_\bullet \to \mathcal{Y}$ and replace $\mathcal{Y}$ by a quasi-compact derived algebraic space. Take now a smooth presentation $\mathcal{X}_\bullet \to \mathcal{X}$ with $\mathcal{X}_0$ quasi-compact. Then the spectral sequence in the above Lemma implies that it is enough to show the result for each morphism $\mathcal{X}_n \to \mathcal{Y}$. Thus we may assume that $\mathcal{X}$ and $\mathcal{Y}$ are quasi-compact derived algebraic spaces. Since Definition 5.4.3 of constructible sheaves only depends on the truncated data, we can reduce to the non-derived setting, where the result is shown in [O1, Proposition 9.9]. □

On the homotopy category we denote the derived functor by
$$\mathrm{R}f_* : \mathrm{D}_{\mathrm{c}}^{(+)}(\mathcal{X}_{\text{lis-et}}) \longrightarrow \mathrm{D}_{\mathrm{c}}^{(+)}(\mathcal{Y}_{\text{lis-et}}).$$

Remark. If we set $\mathcal{X} = \iota(X)$ and $\mathcal{Y} = \iota(Y)$ with X and Y algebraic stacks over k, then we recover the construction in [O1, LO1] of the derived direct image
$$\mathrm{R}f_* : \mathrm{D}_{\mathrm{c}}^{(+)}((\iota X)_{\text{lis-et}}, \Lambda) \longrightarrow \mathrm{D}_{\mathrm{c}}^{(+)}((\iota Y)_{\text{lis-et}}, \Lambda).$$

6.3. **Derived inverse image.** We construction the derived inverse image with the help of simplicial description of cartesian sheaves. Let $f : \mathcal{X} \to \mathcal{Y}$ a morphism of geometric derived stacks which are locally of finite presentation. Take an n-atlas $\{Y_i\}_{i\in I}$ of $\mathcal{Y}$ with some n, and denote by $\mathcal{Y}_0 = \coprod_{i\in I} Y_i \to \mathcal{Y}$ the natural surjection. Let us also take an n-atlas $\{X_j\}_{j\in J}$ of the fiber product $\widetilde{\mathcal{X}} := \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}_0$ and denote $\mathcal{X}_0 := \coprod_{j\in J} X_j \to \widetilde{\mathcal{X}}$. Thus we have a square dSt_k:

$$\begin{array}{ccc} \widetilde{\mathcal{X}} & \longrightarrow & \mathcal{X} \\ \big\downarrow & & \big\downarrow{\scriptstyle f} \\ \mathcal{Y}_0 & \longrightarrow & \mathcal{Y} \end{array}$$

Let $e_{\mathcal{X}} : \mathcal{X}_\bullet \to \widetilde{\mathcal{X}}$ and $e_{\mathcal{Y}} : \mathcal{Y}_\bullet \to \mathcal{Y}$ be the smooth presentations with $\mathcal{X}_0 := \coprod_{j\in J} X_j$ and $\mathcal{Y}_0 = \coprod_{i\in I} Y_i$. The morphism $f : \mathcal{X} \to \mathcal{Y}$ induces a morphism $\mathcal{X}_0 \to \mathcal{Y}_0$ of derived algebraic spaces, and it further induces a morphism $f_\bullet : \mathcal{X}_\bullet \to \mathcal{Y}_\bullet$ of simplicial derived algebraic spaces. Restricting to $\mathbf{\Delta}^{\text{str}} \subset \mathbf{\Delta}$, we have the following square of strictly simplicial derived stacks:

$$\begin{array}{ccc} \mathcal{X}_\bullet^{\text{str}} & \xrightarrow{e_{\mathcal{X}}} & \mathcal{X} \\ {\scriptstyle f_\bullet}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ \mathcal{Y}_\bullet^{\text{str}} & \xrightarrow[e_{\mathcal{Y}}]{} & \mathcal{Y} \end{array}$$

The morphism $f_\bullet$ induces a geometric morphism
$$f_{\bullet,\text{et}} : \mathcal{X}^{\text{str}}_{\bullet,\text{et}} \longrightarrow \mathcal{Y}^{\text{str}}_{\bullet,\text{et}}$$
of ∞-topoi, which induces a functor
$$f_\bullet^* : \mathsf{Mod}_{\mathrm{c}}(\mathcal{Y}^{\text{str}}_{\bullet,\text{et}}) \longrightarrow \mathsf{Mod}_{\mathrm{c}}(\mathcal{X}^{\text{str}}_{\bullet,\text{et}}).$$
Here we denoted $\mathsf{Mod}_{\mathrm{c}}(\mathcal{Y}^{\text{str}}_{\bullet,\text{et}}) := \mathsf{Mod}_{\mathrm{c}}(\mathcal{Y}^{\text{str}}_{\bullet,\text{et}}, \Lambda)$. On the other hand, by Proposition 5.4.6, we have equivalences
$$r_{\mathcal{X}} : \mathsf{Mod}_{\mathrm{c}}(\mathcal{X}_{\text{lis-et}}) \xrightarrow{\sim} \mathsf{Mod}_{\mathrm{c}}(\mathcal{X}^{\text{str}}_{\bullet,\text{et}}), \quad r_{\mathcal{Y}} : \mathsf{Mod}_{\mathrm{c}}(\mathcal{Y}_{\text{lis-et}}) \xrightarrow{\sim} \mathsf{Mod}_{\mathrm{c}}(\mathcal{Y}^{\text{str}}_{\bullet,\text{et}}).$$

Definition. We define

$$f^* : \mathsf{Mod}_{\mathrm{c}}(\mathcal{Y}_{\text{lis-et}}) \longrightarrow \mathsf{Mod}_{\mathrm{c}}(\mathcal{X}_{\text{lis-et}})$$

to be the composition $r_{\mathcal{X}}^{-1} \circ f_\bullet^* \circ r_{\mathcal{Y}}$, and call it the *inverse image functor.*

The functor f^* is independent of the choice of n-atlases $\{Y_i\}_{i \in I}$ and $\{X_j\}_{j \in J}$ up to contractible ambiguity.

Lemma 6.3.1. *f^* is a left adjoint of the functor $f_* : \mathsf{Mod}_{\mathrm{c}}(\mathcal{X}_{\text{lis-et}}) \to \mathsf{Mod}_{\mathrm{c}}(\mathcal{Y}_{\text{lis-et}})$, and moreover f^* is left exact in the sense of Definition B.10.1.*

Proof. Since on the simplicial level we have a geometric morphism $f_{\bullet,\mathrm{et}} : \mathcal{X}^{\mathrm{str}}_{\bullet,\mathrm{et}} \to \mathcal{Y}^{\mathrm{str}}_{\bullet,\mathrm{et}}$, the induced functor $f_\bullet^*$ actually sits in an adjunction

$$f_\bullet^* : \mathsf{Mod}_{\mathrm{c}}(\mathcal{Y}^{\mathrm{str}}_{\bullet,\mathrm{et}}) \longrightarrow \mathsf{Mod}_{\mathrm{c}}(\mathcal{X}^{\mathrm{str}}_{\bullet,\mathrm{et}}) : f_{\bullet,*},$$

and $f_{\bullet,*}$ is equivalent to the functor induced by the direct image functor f_*^{et}. Thus we have the conclusion. □

The same argument works for the derived ∞-category. Namely, the geometric morphism $f_{\bullet,\mathrm{et}} : \mathcal{X}^{\mathrm{str}}_{\bullet,\mathrm{et}} \to \mathcal{Y}^{\mathrm{str}}_{\bullet,\mathrm{et}}$ induces an adjunction

$$f_\bullet^* : \mathsf{D}_{\infty,\mathrm{c}}(\mathcal{Y}^{\mathrm{str}}_{\bullet,\mathrm{et}}) \rightleftarrows \mathsf{D}_{\infty,\mathrm{c}}(\mathcal{X}^{\mathrm{str}}_{\bullet,\mathrm{et}}) : f_{\bullet,*},$$

and by Theorem 5.4.8 we have also equivalences

$$r_{\mathcal{X}} : \mathsf{D}_{\infty,\mathrm{c}}(\mathcal{X}_{\text{lis-et}}) \xrightarrow{\sim} \mathsf{D}_{\infty,\mathrm{c}}(\mathcal{X}^{\mathrm{str}}_{\bullet,\mathrm{et}}), \quad r_{\mathcal{Y}} : \mathsf{D}_{\infty,\mathrm{c}}(\mathcal{Y}_{\text{lis-et}}) \xrightarrow{\sim} \mathsf{D}_{\infty,\mathrm{c}}(\mathcal{Y}^{\mathrm{str}}_{\bullet,\mathrm{et}}).$$

Definition 6.3.2. We define

$$f^* : \mathsf{D}_{\infty,\mathrm{c}}(\mathcal{Y}_{\text{lis-et}}) \longrightarrow \mathsf{D}_{\infty,\mathrm{c}}(\mathcal{X}_{\text{lis-et}})$$

to be the composition $r_{\mathcal{X}}^{-1} \circ f_\bullet^* \circ r_{\mathcal{Y}}$, and call it the *derived inverse image functor.*

Obviously we have that the derived functor $f^* : \mathsf{D}_{\infty,\mathrm{c}}(\mathcal{Y}_{\text{lis-et}}) \to \mathsf{D}_{\infty,\mathrm{c}}(\mathcal{X}_{\text{lis-et}})$ is a t-exact extension of the functor $f^* : \mathsf{Mod}_{\mathrm{c}}(\mathcal{Y}_{\text{lis-et}}, \Lambda) \to \mathsf{Mod}_{\mathrm{c}}(\mathcal{X}_{\text{lis-et}}, \Lambda)$. On the homotopy category we denote the derived functor by

$$\mathrm{L}f^* : \mathrm{D}_{\mathrm{c}}(\mathcal{Y}_{\text{lis-et}}) \longrightarrow \mathrm{D}_{\mathrm{c}}(\mathcal{X}_{\text{lis-et}}).$$

Remark. If we set $\mathcal{X} = \iota(X)$ and $\mathcal{Y} = \iota(Y)$ with X and Y algebraic stacks over k, then we recover the construction in [O1, LO1] of the derived inverse image

$$\mathrm{L}f_* : \mathrm{D}_{\mathrm{c}}((\iota X)_{\text{lis-et}}, \Lambda) \longrightarrow \mathrm{D}_{\mathrm{c}}((\iota Y)_{\text{lis-et}}, \Lambda).$$

For later use, we record

Lemma 6.3.3. *Let $\mathcal{X}$ and $\mathcal{Y}$ be geometric derived stacks locally of finite presentation, and $f : \mathcal{X} \to \mathcal{Y}$ be a smooth morphism of relative dimension d. Then $f^*\Omega_{\mathcal{Y}} \simeq \Omega_{\mathcal{X}}\langle -d \rangle$*

Proof. Take a smooth presentation $\mathcal{Y}_\bullet \to \mathcal{Y}$ and consider $\widetilde{\mathcal{X}} := \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}_0$. Then we have a diagram

$$\begin{array}{ccccc} \mathcal{X}_0 & \longrightarrow & \widetilde{\mathcal{X}} & \longrightarrow & \mathcal{X} \\ & \searrow & \downarrow & & \downarrow f \\ & & \mathcal{Y}_0 & \longrightarrow & \mathcal{Y} \end{array}$$

of derived stacks. By the assumption on f, the morphism $\mathcal{X}_0 \to \mathcal{Y}_0$ is smooth. Consider the restrictions of $f^*\Omega_{\mathcal{Y}}$ and $\Omega_{\mathcal{X}}\langle -d \rangle$ to $\mathcal{X}_{\text{lis-et}}|_{\mathcal{X}_0}$ coincide and have zero negative $\mathscr{E}xt$'s. Then Lemma 3.7.10 implies the consequence. □

6.4. **Derived internal Hom functor.** Recall that in §3.4 we discussed the internal Hom functor

$$\mathscr{H}om_{\mathcal{A}}(-,-) : \mathsf{Mod}^{\mathrm{stab}}_{\mathcal{A}}(\mathsf{T})^{\mathrm{op}} \times \mathsf{Mod}^{\mathrm{stab}}_{\mathcal{A}}(\mathsf{T}) \longrightarrow \mathsf{Mod}^{\mathrm{stab}}_{\mathcal{A}}(\mathsf{T})$$

for a ringed ∞-topos $(\mathsf{T}, \mathcal{A})$. Applying it to $(\mathsf{T}, \mathcal{A}) = (\mathcal{X}_{\text{lis-et}}, \Lambda)$, we have

Notation. For a geometric derived stack $\mathcal{X}$. we denote the internal Hom functor on $\mathsf{D}_\infty(\mathcal{X}_{\text{lis-et}})$ by

$$\mathscr{H}om(-,-) : \mathsf{D}_\infty(\mathcal{X}_{\text{lis-et}})^{\mathrm{op}} \times \mathsf{D}(\mathcal{X}_{\text{lis-et}}) \longrightarrow \mathsf{D}_\infty(\mathcal{X}_{\text{lis-et}}).$$

We also denote it by $\mathscr{H}om_{\mathcal{X}_{\text{lis-et}}}$ to emphasize the dependence on $\mathcal{X}_{\text{lis-et}}$. The associated functor on the homotopy category is denoted by

$$\mathscr{R}hom(-,-) : \mathrm{D}(\mathcal{X}_{\text{lis-et}})^{\mathrm{op}} \times \mathrm{D}(\mathcal{X}_{\text{lis-et}}) \longrightarrow \mathrm{D}(\mathcal{X}_{\text{lis-et}}).$$

As for the constructible objects, we have

Lemma. Let $\mathcal{X}$ be a geometric derived stack locally of finite presentation. Then for $\mathcal{M} \in \mathsf{D}^{(-)}_{\infty,\mathrm{c}}(\mathcal{X}_{\text{lis-et}})$ and $\mathcal{N} \in \mathsf{D}^{(+)}_{\infty,\mathrm{c}}(\mathcal{X}_{\text{lis-et}})$, we have $\mathscr{H}om(\mathcal{M}, \mathcal{N}) \in \mathsf{D}^{(+)}_{\infty,\mathrm{c}}(\mathcal{X}_{\text{lis-et}})$.

Proof. We follow the argument in [LO1, 4.2.1. Lemma, 4.2.2. Corollary]. For $\mathcal{M}, \mathcal{N} \in \mathsf{D}_{\infty,\mathrm{c}}(\mathcal{X}_{\text{lis-et}})$ and $\mathcal{U} \in \mathsf{dAS}^{\text{lis}}_{\mathcal{X}}$, we have a functorial equivalence

$$\mathscr{H}om_{\mathcal{X}_{\text{lis-et}}}(\mathcal{M}, \mathcal{N})|_{\mathcal{U}_{\text{et}}} \simeq \mathscr{H}om_{\mathcal{U}_{\text{et}}}(\mathcal{M}_{\mathcal{U}}, \mathcal{N}_{\mathcal{U}}).$$

Indeed, for the geometric morphism $\varepsilon : \mathcal{X}_{\text{lis-et}}|_{\mathcal{U}} \to \mathcal{U}_{\text{et}}$ in Lemma 5.5.6, the counit transformation $\varepsilon^*\varepsilon_*\mathcal{M} \to \mathcal{M}$ and $\varepsilon^*\varepsilon_*\mathcal{N} \to \mathcal{N}$ are equivalences in $\mathsf{D}_\infty(\mathcal{X}_{\text{lis-et}}|_{\mathcal{U}})$ since $\mathcal{M}$ and $\mathcal{N}$ have constructible homotopy groups and by by Proposition 3.7.8. On the other hand, the unit transformation $\mathrm{id} \to \varepsilon_*\varepsilon^*$ is an equivalence, so we have

$$\begin{aligned}\mathscr{H}om_{\,\mathcal{X}_{\text{lis-et}}|_{\mathcal{U}}}(\mathcal{M}, \mathcal{N}) &\simeq \mathscr{H}om_{\,\mathcal{X}_{\text{lis-et}}|_{\mathcal{U}}}(\varepsilon^*\varepsilon_*\mathcal{M}, \varepsilon^*\varepsilon_*\mathcal{N}) \simeq \mathscr{H}om_{\mathcal{U}_{\text{et}}}(\varepsilon^*\varepsilon_*\mathcal{M}, \varepsilon^*\varepsilon_*\mathcal{N}) \\ &\simeq \mathscr{H}om_{\mathcal{U}_{\text{et}}}(\varepsilon_*\mathcal{M}, \varepsilon_*\varepsilon^*\varepsilon_*\mathcal{N}) \simeq \mathscr{H}om_{\mathcal{U}_{\text{et}}}(\varepsilon_*\mathcal{M}, \varepsilon_*\mathcal{N}) \simeq \mathscr{H}om_{\mathcal{U}_{\text{et}}}(\mathcal{M}_{\mathcal{U}}, \mathcal{N}_{\mathcal{U}}).\end{aligned}$$

Now consider the following triangle with f a morphism in $\mathsf{dAS}^{\text{lis,fp}}_{\mathcal{X}}$.

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{\ f\ } & \mathcal{U} \\ & {\scriptstyle B}\searrow \quad \swarrow{\scriptstyle A} & \\ & \mathcal{X} & \end{array}$$

Recall the functor $f^* : \mathsf{D}_{\infty,\mathrm{c}}(\mathcal{U}_{\text{et}}) \to \mathsf{D}_{\infty,\mathrm{c}}(\mathcal{V}_{\text{et}})$ (Definition 6.3.2). Then for $\mathcal{M} \in \mathsf{D}^{(-)}_{\infty,\mathrm{c}}(\mathcal{X}_{\text{lis-et}})$ and $\mathcal{N} \in \mathsf{D}^{(+)}_{\infty,\mathrm{c}}(\mathcal{X}_{\text{lis-et}})$, we have $\mathcal{M}_{\mathcal{U}} \in \mathsf{D}^{-}_{\infty,\mathrm{c}}(\mathcal{U}_{\text{et}})$, $\mathcal{M}_{\mathcal{V}} \in \mathsf{D}^{-}_{\infty,\mathrm{c}}(\mathcal{V}_{\text{et}})$, $\mathcal{N}_{\mathcal{U}} \in \mathsf{D}^{+}_{\infty,\mathrm{c}}(\mathcal{U}_{\text{et}})$ and $\mathcal{N}_{\mathcal{V}} \in \mathsf{D}^{+}_{\infty,\mathrm{c}}(\mathcal{V}_{\text{et}})$. We also have a morphism

$$f_*\mathscr{H}om_{\mathcal{U}_{\text{et}}}(\mathcal{M}_{\mathcal{U}}, \mathcal{N}_{\mathcal{U}}) \longrightarrow \mathscr{H}om_{\mathcal{V}_{\text{et}}}(f^*\mathcal{M}, f^*\mathcal{N}) \simeq \mathscr{H}om_{\mathcal{V}_{\text{et}}}(\mathcal{M}_{\mathcal{V}}, \mathcal{N}_{\mathcal{V}}).$$

The consequence holds if we show this morphism is an equivalence. But we have

$$f^*\mathscr{H}om_{\mathcal{U}_{\text{et}}}(\mathcal{M}_{\mathcal{U}}, \mathcal{N}_{\mathcal{U}}) \simeq f^*A^*\mathscr{H}om_{\mathcal{X}_{\text{lis-et}}}(\mathcal{M}, \mathcal{N}) \simeq B^*\mathscr{H}om_{\mathcal{X}_{\text{lis-et}}}(\mathcal{M}, \mathcal{N}) \simeq \mathscr{H}om_{\mathcal{V}_{\text{et}}}(\mathcal{M}_{\mathcal{V}}, \mathcal{N}_{\mathcal{V}}).$$

□

Let $\mathcal{X}_\bullet \to \mathcal{X}$ be a smooth presentation of a geometric derived stack $\mathcal{X}$, and $\mathcal{X}_{\text{lis-et}}|_{\mathcal{X}^{\text{str}}_\bullet}$ be the strictly simplicial ∞-topos obtained by restricting the lisse-étale ∞-topos to the strictly simplicial derived algebraic space $\mathcal{X}^{\text{str}}_\bullet$. We have the associated geometric morphism denote by $p : \mathcal{X}_{\text{lis-et}}|_{\mathcal{X}^{\text{str}}_\bullet} \to \mathcal{X}_{\text{lis-et}}$. Then we have

Lemma. For $\mathcal{M}, \mathcal{N} \in \mathsf{D}_{\infty,\mathrm{c}}(\mathcal{X}_{\text{lis-et}})$ we have a functorial equivalence

$$p^*\mathscr{H}om_{\mathcal{X}_{\text{lis-et}}}(\mathcal{M}, \mathcal{N}) \simeq \mathscr{H}om_{\,\mathcal{X}_{\text{lis-et}}|_{\mathcal{X}^{\text{str}}_\bullet}}(p^*\mathcal{M}, p^*\mathcal{N})$$

Proof. We follow the argument in [LO1, 4.2.4. Lemma]. We can take an equivalence $\mathcal{N} \to \mathcal{I}$ in $\mathsf{D}_\infty(\mathcal{X}_{\text{lis-et}})$ with $\mathcal{I}$ having injective homotopy groups. Then we have

$$p^*\mathscr{H}om_{\mathcal{X}_{\text{lis-et}}}(\mathcal{M}, \mathcal{N}) \simeq p^*\mathscr{H}om_{\mathcal{X}_{\text{lis-et}}}(\mathcal{M}, \mathcal{I}) \simeq \mathscr{H}om_{\,\mathcal{X}_{\text{lis-et}}|_{\mathcal{X}^{\text{str}}_\bullet}}(p^*\mathcal{M}, p^*\mathcal{I}).$$

We can also take an equivalence $p^*\mathcal{I} \to \mathcal{J}_\bullet$ in $\mathsf{D}_\infty(\mathcal{X}_{\text{lis-et}}|_{\mathcal{X}^{\text{str}}_\bullet})$, where $\mathcal{J}_\bullet$ has injective homotopy groups. This equivalence induces a morphism

$$\mathscr{H}om_{\,\mathcal{X}_{\text{lis-et}}|_{\mathcal{X}^{\text{str}}_\bullet}}(p^*\mathcal{M}, p^*\mathcal{I}) \to \mathscr{H}om_{\,\mathcal{X}_{\text{lis-et}}|_{\mathcal{X}^{\text{str}}_\bullet}}(p^*\mathcal{M}, \mathcal{J}_\bullet).$$

On the other hand, we have $\mathscr{H}om_{\,\mathcal{X}_{\text{lis-et}}|_{\mathcal{X}^{\text{str}}_\bullet}}(p^*\mathcal{M}, p^*\mathcal{N}) \xrightarrow{\sim} \mathscr{H}om_{\,\mathcal{X}_{\text{lis-et}}|_{\mathcal{X}^{\text{str}}_\bullet}}(p^*\mathcal{M}, \mathcal{J}_\bullet)$. Composing these morphisms, we have $p^*\mathscr{H}om_{\mathcal{X}_{\text{lis-et}}}(\mathcal{M}, \mathcal{N}) \to \mathscr{H}om_{\,\mathcal{X}_{\text{lis-et}}|_{\mathcal{X}_\bullet}}(p^*\mathcal{M}, p^*\mathcal{N})$. It is enough to show that this morphism is an equivalence. For that, it suffices to show that each morphism $\mathscr{H}om_{\,\mathcal{X}_{\text{lis-et}}|_{\mathcal{X}^{\text{str}}_n}}(p_n^*\mathcal{M}, p_n^*\mathcal{I}) \to \mathscr{H}om_{\,\mathcal{X}_{\text{lis-et}}|_{\mathcal{X}_n}}(p_n^*\mathcal{M}, \mathcal{J}_n)$ is an equivalence in $\mathsf{D}_\infty(\mathcal{X}_{\text{lis-et}}|_{\mathcal{X}_n})$, where $p_n : \mathcal{X}_{\text{lis-et}}|_{\mathcal{X}_n} \to \mathcal{X}_{\text{lis-et}}$ is the geometric morphism associated to the localization. The last claim can be checked by routine (see also the last part of the proof of [LO1, 4.2.4. Lemma]). □

Using this Lemma, we can show the following restatement of [LO1, 4.2.3. Proposition].

Lemma 6.4.1. Let $\mathcal{X}_\bullet \to \mathcal{X}$ be a smooth presentation of a geometric stack $\mathcal{X}$, We denote by $\mathcal{M}_{\text{et}}, \mathcal{N}_{\text{et}} \in \mathsf{D}_{\infty,\mathrm{c}}(\mathcal{X}^{\text{str}}_{\bullet,\text{et}})$ the restrictions of $\mathcal{M}, \mathcal{N}$ to the étale ∞-topos of $\mathcal{X}^{\text{str}}_\bullet$. Then we have a functorial equivalence

$$\mathscr{H}om_{\mathcal{X}_{\text{lis-et}}}|_{(\mathcal{M},\mathcal{N})}\, \mathcal{X}^{\text{str}}_{\bullet,\text{et}} \xrightarrow{\sim} \mathscr{H}om_{\mathcal{X}^{\text{str}}_{\bullet,\text{et}}}(\mathcal{M}_{\text{et}}, \mathcal{N}_{\text{et}}).$$

Proof. We follow the proof of [LO1, 4.2.3. Proposition]. Let us denote by $e : \mathcal{X}_{\text{lis-et}}|_{\mathcal{X}^{\text{str}}_{\bullet}} \to \mathcal{X}^{\text{str}}_{\bullet,\text{et}}$ the geometric morphism of ∞-topoi induced by $\mathrm{Et}(\mathcal{X}^{\text{str}}_{\bullet}) \hookrightarrow \mathrm{LE}(\mathcal{X}^{\text{str}}_{\bullet})$. Then we have $\mathcal{M}_{\text{et}} = e_* p^* \mathcal{M}$, $\mathcal{N}_{\text{et}} = e_* p^* \mathcal{N}$ and $\mathscr{H}om_{\mathcal{X}_{\text{lis-et}}}(\mathcal{M},\mathcal{N})|_{\mathcal{X}^{\text{str}}_{\bullet,\text{et}}} = e_* p^* \mathscr{H}om_{\mathcal{X}_{\text{lis-et}}}(\mathcal{M},\mathcal{N})$ by definition. We also have equivalences $e^*\mathcal{M}_{\text{et}} \xrightarrow{\sim} \pi^*\mathcal{M}$ and $e^*\mathcal{N}_{\text{et}} \xrightarrow{\sim} \pi^*\mathcal{N}$ since the counit transformation $e^* e_* \to \mathrm{id}$ is an equivalence by Proposition 3.7.8. Then

$$\begin{aligned}\mathscr{H}om_{\mathcal{X}_{\text{lis-et}}}(\mathcal{M},\mathcal{N})|_{\mathcal{X}^{\text{str}}_{\bullet,\text{et}}} &= e_* p^* \mathscr{H}om_{\mathcal{X}_{\text{lis-et}}}(\mathcal{M},\mathcal{N}) \simeq e_* \mathscr{H}om_{\mathcal{X}_{\text{lis-et}}|_{\mathcal{X}^{\text{str}}_{\bullet}}}(p^*\mathcal{M}, p^*\mathcal{N}) \\ &\simeq e_* \mathscr{H}om_{\mathcal{X}_{\text{lis-et}}|_{\mathcal{X}^{\text{str}}_{\bullet}}}(e_*\mathcal{M}_{\text{et}}, e_*\mathcal{N}_{\text{et}}) \simeq \mathscr{H}om_{\mathcal{X}_{\text{et},\bullet}}(\mathcal{M}_{\text{et}}, \mathcal{N}_{\text{et}}).\end{aligned}$$

□

Now we have the following type of projection formula. The proof is the same as in [LO1, 4.3.1. Proposition] with the help of Lemma 6.4.1, so we omit it.

Proposition 6.4.2. Let $\mathcal{X}$, $\mathcal{Y}$ be geometric derived stacks locally of finite presentation, and $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of of finite presentation. Then, for $\mathcal{M} \in \mathsf{D}_{\infty,c}(\mathcal{Y}_{\text{lis-et}})$ and $\mathcal{N} \in \mathsf{D}^{(+)}_{\infty,c}(\mathcal{X}_{\text{lis-et}})$, we have a functorial equivalence

$$\mathscr{H}om(\mathcal{M}, f_*\mathcal{N}) \xrightarrow{\sim} f_* \mathscr{H}om(f^*\mathcal{M}, \mathcal{N})$$

in $\mathsf{D}_{\infty,c}(\mathcal{Y}_{\text{lis-et}})$.

For later use, we record the following lemma. Recall the dualizing object $\Omega_{\mathcal{X}}$ for a derived stack $\mathcal{X}$ (Definition 5.5.9).

Lemma 6.4.3. Let $\mathcal{X}$, $\mathcal{Y}$ and $f : \mathcal{X} \to \mathcal{Y}$ be as Proposition 6.4.2. For $\mathcal{M} \in \mathsf{D}_{\infty,c}(\mathcal{Y}_{\text{lis-et}})$, the canonical morphism

$$f^* \mathscr{H}om(\mathcal{M}, \Omega_{\mathcal{Y}}) \longrightarrow \mathscr{H}om(f^*\mathcal{M}, f^*\Omega_{\mathcal{Y}})$$

is an equivalence.

6.5. **Derived tensor functor.** Let $\mathcal{X}$ be a geometric derived stack locally of finite presentation. In the rest part of this section, we denote by $\otimes$ the tensor functor $\otimes_{\Lambda}$ on $\mathsf{D}_{\infty}(\mathcal{X}_{\text{lis-et}})$. Recall also the dualizing functor $\mathrm{D}_{\mathcal{X}} : \mathsf{D}_{\infty}(\mathcal{X}_{\text{lis-et}}) \to \mathsf{D}_{\infty}(\mathcal{X}_{\text{lis-et}})^{\text{op}}$ (Notation 5.5.12).

Lemma 6.5.1. (1) For $\mathcal{M},\mathcal{N} \in \mathsf{D}_{\infty,c}(\mathcal{X}_{\text{lis-et}})$, we have $\mathscr{H}om(\mathcal{M},\mathcal{N}) \simeq \mathrm{D}_{\mathcal{X}}(\mathcal{M} \otimes \mathrm{D}_{\mathcal{X}}(\mathcal{N}))$.
(2) For $\mathcal{M},\mathcal{N} \in \mathsf{D}^{(-)}_{\infty,c}(\mathcal{X}_{\text{lis-et}})$, we have $\mathcal{M} \otimes \mathcal{N} \in \mathsf{D}^{(-)}_{\infty,c}(\mathcal{X}_{\text{lis-et}})$.

Proof. The argument in [LO1, 4.5.1 Lemma] works with the help of Proposition 5.5.13. □

6.6. **Shriek functors.** Following the ordinary algebraic stack case [LO1, §4.6], we introduce the derived functors $f_!$ and $f^!$ using the dualizing functor $\mathrm{D}_{\mathcal{X}}$.

Definition 6.6.1. Let $\mathcal{X}$ and $\mathcal{Y}$ be geometric derived stacks locally of finite presentation, and let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of finite presentation. We define the functor $f_!$ by

$$f_! := \mathrm{D}_{\mathcal{Y}} \circ f_* \circ \mathrm{D}_{\mathcal{X}} : \mathsf{D}^{(-)}_{\infty,c}(\mathcal{X}_{\text{lis-et}}) \longrightarrow \mathsf{D}^{(+)}_{\infty,c}(\mathcal{Y}_{\text{lis-et}}).$$

Here f_* is the derived direct image functor (Proposition 6.2.1). We also define the functor $f^!$ by

$$f^! := \mathrm{D}_{\mathcal{X}} \circ f^* \circ \mathrm{D}_{\mathcal{Y}} : \mathsf{D}_{\infty,c}(\mathcal{Y},\Lambda) \longrightarrow \mathsf{D}_{\infty,c}(\mathcal{X},\Lambda),$$

where f^* is the derived inverse image functor (Definition 6.3.2).

We immediately have

Proposition 6.6.2. Let $\mathcal{X}$, $\mathcal{Y}$ and $f : \mathcal{X} \to \mathcal{Y}$ be as in Definition 6.6.1. For $\mathcal{M} \in \mathsf{D}^{(-)}_{\infty,c}(\mathcal{X}_{\text{lis-et}})$ and $\mathcal{N} \in \mathsf{D}_{\infty,c}(\mathcal{Y}_{\text{lis-et}})$, we have a functorial equivalence

$$f_* \mathscr{H}om(\mathcal{M}, f^!\mathcal{N}) \simeq \mathscr{H}om(f_!\mathcal{M}, \mathcal{N}).$$

Proof. The statement follows from the sequence of equivalences

$$\begin{aligned} f_* \mathscr{H}om(\mathcal{M}, f^!\mathcal{N}) &\overset{(*1)}{\simeq} f_* \mathscr{H}om(\mathrm{D}_{\mathcal{X}} f^!\mathcal{N}, \mathrm{D}_{\mathcal{X}} \mathcal{M}) \overset{(*2)}{\simeq} f_* \mathscr{H}om(f^* \mathrm{D}_{\mathcal{Y}} \mathcal{N}, \mathrm{D}_{\mathcal{X}} \mathcal{M}) \\ &\overset{(*3)}{\simeq} \mathscr{H}om(\mathrm{D}_{\mathcal{Y}} \mathcal{N}, f_* \mathrm{D}_{\mathcal{X}} \mathcal{M}) \overset{(*1)}{\simeq} \mathscr{H}om(\mathrm{D}_{\mathcal{Y}} f_* \mathrm{D}_{\mathcal{X}} \mathcal{M}, \mathrm{D}_{\mathcal{Y}} \mathrm{D}_{\mathcal{Y}} \mathcal{N}) \overset{(*4)}{\simeq} \mathscr{H}om(f_!\mathcal{M}, \mathcal{N}). \end{aligned}$$

Here $(*1)$ is by Proposition 5.5.13 (3), $(*2)$ is by Definition 6.6.1, $(*3)$ is by Proposition 6.4.2, and $(*4)$ is by Definition 6.6.1 and Proposition 5.5.13 (2). □

We also have the projection formula.

Proposition 6.6.3. Let $f : \mathcal{X} \to \mathcal{Y}$ be of finite type.

(1) For $\mathcal{M} \in \mathsf{D}^{(-)}_{\infty,c}(\mathcal{X}_{\text{lis-et}})$ and $\mathcal{N} \in \mathsf{D}^{(-)}_{\infty,c}(\mathcal{Y}_{\text{lis-et}})$ we have a functorial equivalence

$$f_!(\mathcal{M} \otimes f^*\mathcal{N}) \simeq (f_!\mathcal{M}) \otimes \mathcal{N}.$$

(2) For $\mathcal{M} \in \mathsf{D}^{(-)}_{\infty,c}(\mathcal{X}_{\text{lis-et}})$ and $\mathcal{N} \in \mathsf{D}^{(+)}_{\infty,c}(\mathcal{Y}_{\text{lis-et}})$ we have a functorial equivalence

$$f^! \mathscr{H}om(\mathcal{M}, \mathcal{N}) \simeq \mathscr{H}om(f^*\mathcal{M}, f^!\mathcal{N}).$$

Proof. The argument in [LO1, 4.5.2, 4.5.3] with Lemma 6.5.1 works. □

Let us give a few properties of the functors $f^!$ and $f_!$. The proof is now by a standard argument, so we omit it.

Lemma 6.6.4. Let $\mathcal{X}$ and $\mathcal{Y}$ be geometric derived stacks locally of finite presentation.

(1) If $f : \mathcal{X} \to \mathcal{Y}$ is a smooth morphism of relative dimension d, then $f^! = f^*\langle d\rangle$.
(2) If $j : \mathcal{X} \to \mathcal{Y}$ is an open immersion, then $j^! = j^*$ and $j_!$ is equivalent to the extension by zero functor (see §4.4.1). In particular, we have a morphism $j_! \to j_*$.

In the rest part of this subsection, we discuss base change theorems for lisse-étale sheaves of derived stacks under several situations. Let us consider a cartesian square

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{\pi} & \mathcal{X} \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{Y}' & \xrightarrow{p} & \mathcal{Y} \end{array}$$

of derived stacks locally of finite presentation with f of finite presentation. We have a morphism

$$p^* f_! \longrightarrow \varphi_! \pi^*$$

in $\mathsf{Fun}(\mathsf{D}^{(-)}_{\infty,c}(\mathcal{X}_{\text{lis-et}}), \mathsf{D}^{(-)}_{\infty,c}(\mathcal{Y}'_{\text{lis-et}}))$, and

$$p^! f_* \longrightarrow \phi_* \pi^!$$

in $\mathsf{Fun}(\mathsf{D}^{(+)}_{\infty,c}(\mathcal{X}_{\text{lis-et}}), \mathsf{D}^{(+)}_{\infty,c}(\mathcal{Y}'_{\text{lis-et}}))$.

Proposition 6.6.5. If p is smooth, then the morphisms $p^* f_! \to \varphi_! \pi^*$ and $p^! f_* \to \phi_* \pi^!$ are equivalence.

Proof. so that the functor p^* is defined on all $\mathsf{D}_{\infty,c}(\mathcal{Y}_{\text{lis-et}})$ and equal to the restriction from $\mathcal{Y}_{\text{lis-et}}$ to $\mathcal{Y}'_{\text{lis-et}}$. □

We expect to have simple extensions of the base change theorems for algebraic stacks shown in [LO1, §5], but we will not pursuit them.

7. Adic coefficient case

In this section we give an extension of the results in the previous section to the adic coefficient case.

Let k be a fixed commutative ring, and Λ be a complete discrete valuation ring with characteristic ℓ. We assume that ℓ is invertible in k. We write $\Lambda = \varprojlim_n \Lambda_n$, $\Lambda_n := \Lambda/\mathfrak{m}^n$ with $\mathfrak{m} \subset \Lambda$ the maximal ideal. We denote by $\Lambda_\bullet = (\Lambda_n)_{n\in\mathbb{N}}$ the projective system of commutative rings.

7.1. Projective systems of ringed ∞-topoi.

Let T be an ∞-topos. Recall Definition 3.7.4 of the ∞-topos of projective systems in T. It is an I-simplicial ∞-topos $\mathsf{T}^{\mathbb{N}}$ with $\mathsf{I} = \mathsf{N}(\mathbb{N})$, equipped with a geometric morphism $e_n : \mathsf{T} \to \mathsf{T}^{\mathbb{N}}$ of ∞-topoi for each $n \in \mathbb{N}$. The adjunction

$$e_n^{-1} : \mathsf{T}^{\mathbb{N}} \rightleftarrows \mathsf{T} : e_{n,*}.$$

associated to e_n is described as follows. For $U_\bullet \in \mathsf{T}^{\mathbb{N}}$, we have $e_n^{-1} U_\bullet = U_n$. For $U \in \mathsf{T}$, we have

$$(e_{n,*}U)_m = \begin{cases} U & (m \geq n), \\ * & (m < n). \end{cases}$$

Here $*$ denotes a final object of $\mathcal{S}$. We also have a geometric morphism $p : \mathsf{T}^{\mathbb{N}} \to \mathsf{T}$ which corresponds to the adjunction

$$p^{-1} : \mathsf{T} \rightleftarrows \mathsf{T}^{\mathbb{N}} : p_*$$

with $p^{-1}(U) = (U)_{n\in\mathbb{N}}$.

Let $R_\bullet = (R_n)_{n\ge 0}$ be a projective system of commutative rings, We ring the ∞-topos T by the constant sheaf R_n, and ring $\mathsf{T}^{\mathbb{N}}$ by $R_\bullet$. Then we have geometric morphisms

$$e_n : (\mathsf{T}, R_n) \longrightarrow (\mathsf{T}^{\mathbb{N}}, R_\bullet), \quad p : (\mathsf{T}^{\mathbb{N}}, R_\bullet) \longrightarrow (\mathsf{T}, R_n)$$

induced by $e_n : \mathsf{T} \to \mathsf{T}^{\mathbb{N}}$ and $p : \mathsf{T}^{\mathbb{N}} \to \mathsf{T}$.

Let us apply this notation for $\mathsf{T} = \mathcal{X}_{\text{lis-et}}$ and $R_\bullet = \Lambda_\bullet$, where $\mathcal{X}$ is a geometric derived stack over k and $\Lambda_\bullet$ is the complete discrete valuation ring in the beginning of this section. Recall the notion of *AR-nullity*:

Notation. A projective system $M_\bullet = (M_n)_{n\in\mathbb{N}}$ in an additive category is *AR-null* if there exists an $r \in \mathbb{Z}$ such that the projection $M_{n+r} \to M_n$ is zero for every n.

Following [LO2, 2.1.1. Definition] we introduce

Definition. Let $\mathcal{M}_\bullet = (\mathcal{M}_n)_{n\in\mathbb{N}}$ be an object of $\mathsf{D}_\infty(\mathcal{X}^{\mathbb{N}}_{\text{lis-et}}, \Lambda_\bullet)$.

(1) $\mathcal{M}_\bullet$ is *AR-null* if the projective system $\pi_j\mathcal{M}_\bullet$ of the j-th homotopy groups is AR-null for any $j \in \mathbb{Z}$.
(2) $\mathcal{M}_\bullet$ is *constructible* if $\pi_j\mathcal{M}_n$ is constructible for any $j \in \mathbb{Z}$ and $n \in \mathbb{N}$.
(3) $\mathcal{M}_\bullet$ is *almost zero* if the restriction $(\pi_j\mathcal{M}_\bullet)|_{\mathcal{U}_{\text{et}}}$ is AR-null for any $\mathcal{U} \in \mathsf{dAS}^{\text{lis}}_{\mathcal{X}}$ and $j \in \mathbb{Z}$.

Following [LO2, 2.2], we give a description of AR-null objects and almost zero objects by the restrictions to étale ∞-topos. Let us take $\mathcal{U} \in \mathsf{dAS}^{\text{lis}}_{\mathcal{X}}$. We denote by $p : \mathcal{X}^{\mathbb{N}}_{\text{lis-et}} \to \mathcal{X}$ the projection, Identifying $\mathcal{U}$ with the constant projective system $p^*\mathcal{U}$, we have $(\mathcal{X}_{\text{lis-et}}|_{\mathcal{U}})^{\mathbb{N}} \simeq (\mathcal{X}^{\mathbb{N}}_{\text{lis-et}})|_{\mathcal{U}}$, which will be denoted by $\mathcal{X}^{\mathbb{N}}_{\text{lis-et}}|_{\mathcal{U}}$. Then we have a square

$$\begin{array}{ccc} \mathcal{X}^{\mathbb{N}}_{\text{lis-et}}|_{\mathcal{U}} & \xrightarrow{j^{\mathbb{N}}} & \mathcal{X}^{\mathbb{N}}_{\text{lis-et}} \\ \downarrow{\scriptstyle p|_{\mathcal{U}}} & & \downarrow{\scriptstyle p} \\ \mathcal{X}_{\text{lis-et}}|_{\mathcal{U}} & \xrightarrow[j]{} & \mathcal{X}_{\text{lis-et}} \end{array}$$

in RTop, where j and $j^{\mathbb{N}}$ are the canonical functors (Corollary B.3.2, Fact 3.1.5). By the exactness of j^* and $(j^{\mathbb{N}})^*$, we have $(p|_{\mathcal{U}})_*(j^{\mathbb{N}})^* \simeq j^*p_*$ in $\mathsf{Fun}(\mathsf{D}_\infty(\mathcal{X}^{\mathbb{N}}_{\text{lis-et}}), \mathsf{D}_\infty(\mathcal{X}_{\text{lis-et}}|_{\mathcal{U}}))$.

On the other hand, denoting by $\varepsilon_{\mathcal{U}} : \mathcal{X}_{\text{lis-et}} \to \mathcal{U}_{\text{et}}$ and $\varepsilon^{\mathbb{N}}_{\mathcal{U}} : \mathcal{X}^{\mathbb{N}}_{\text{lis-et}} \to \mathcal{U}^{\mathbb{N}}_{\text{et}}$ the geometric morphisms induced by the natural embedding $\text{Et}(\mathcal{U}) \to \text{LE}(\mathcal{X})|_{\mathcal{U}}$ (Lemma 5.5.6), we have a square

$$\begin{array}{ccc} \mathcal{X}^{\mathbb{N}}_{\text{lis-et}}|_{\mathcal{U}} & \xrightarrow{\varepsilon^{\mathbb{N}}_{\mathcal{U}}} & \mathcal{U}^{\mathbb{N}}_{\text{et}} \\ \downarrow{\scriptstyle p|_{\mathcal{U}}} & & \downarrow{\scriptstyle p_{\mathcal{U}}} \\ \mathcal{X}_{\text{lis-et}}|_{\mathcal{U}} & \xrightarrow[\varepsilon_{\mathcal{U}}]{} & \mathcal{U}_{\text{et}} \end{array}$$

in RTop. Since $(\varepsilon_{\mathcal{U}})_*$ and $(\varepsilon^{\mathbb{N}}_{\mathcal{U}})_*$ are exact, we have $(p_{\mathcal{U}})_*(\varepsilon^{\mathbb{N}}_{\mathcal{U}})_* \simeq (\varepsilon_{\mathcal{U}})_*(p|_{\mathcal{U}})_*$.

For $\mathcal{M} \in \mathsf{D}_\infty(\mathcal{X}_{\text{lis-et}}, \Lambda)$, we denote by $\mathcal{M}_{\mathcal{U}} \in \mathsf{D}_\infty(\mathcal{U}_{\text{et}}, \Lambda)$ the restriction (Notation 5.2.2). By the identification of $\mathcal{U}$ with $p^*\mathcal{U}$, we can regard $\mathcal{M}$ as an object of $\mathsf{D}_\infty(\mathcal{X}^{\mathbb{N}}_{\text{lis-et}}|_{\mathcal{U}}, \Lambda_\bullet)$, which will be denoted by the same symbol $\mathcal{M}$. Then by the argument above we immediately have

Lemma ([LO2, 2.2.1. Lemma])**.** We have $(p_{\mathcal{U}})_*\mathcal{M}_{\mathcal{U}} \simeq ((p|_{\mathcal{U}})_*\mathcal{M})_{\mathcal{U}}$ in $\mathsf{D}_\infty(\mathcal{U}_{\text{et}}, \Lambda)$.

Now AR-null objects and almost zero objects are described as follows.

Fact 7.1.1 ([LO2, 2.2.2. Proposition])**.** Let $\mathcal{M}_\bullet \in \mathsf{D}_\infty(\mathcal{X}^{\mathbb{N}}_{\text{lis-et}}, \Lambda_\bullet)$.

(1) If $\mathcal{M}_\bullet$ is AR-null, then $p_*\mathcal{M}_\bullet = 0$ in $\mathsf{D}_\infty(\mathcal{X}_{\text{lis-et}}|_{\mathcal{U}}, \Lambda)$.
(2) If $\mathcal{M}_\bullet$ is almost zero, then $p_*\mathcal{M}_\bullet = 0$ in $\mathsf{D}_\infty(\mathcal{X}_{\text{lis-et}}|_{\mathcal{U}}, \Lambda)$.

Definition. Let $\mathcal{M}_\bullet = (\mathcal{M}_n)_{n\in\mathbb{N}}$ be an object of $\mathsf{D}_\infty(\mathcal{X}^{\mathbb{N}}_{\text{lis-et}}, \Lambda_\bullet)$.

(1) $\mathcal{M}_\bullet$ is *adic* if each $\mathcal{M}_n$ is constructible and each morphism $\Lambda_n \otimes_{\Lambda_{n+1}} \mathcal{M}_{n+1} \to \mathcal{M}_n$ is an equivalence.
(2) $\mathcal{M}_\bullet$ is *almost adic* if each $\mathcal{M}_n$ is constructible, and if for each $\mathcal{U} \in \mathsf{dAS}^{\text{lis}}_{\mathcal{X}}$ there is a morphism $\mathcal{N} \to \mathcal{M}_\bullet|_{\mathcal{U}_{\text{et}}}$ from an adic object $\mathcal{N} \in \mathsf{D}_\infty(\mathcal{U}^{\mathbb{N}}_{\text{et}}, \Lambda_\bullet)$ with almost zero kernel and cokernel in the homotopy category.
(3) $\mathcal{M}_\bullet$ is a *λ-object* if $p_j\mathcal{M}_\bullet$ is almost adic for each $j \in \mathbb{Z}$. The full sub-∞-category of $\mathsf{D}(\mathcal{X}^{\mathbb{N}}_{\text{lis-et}}, \Lambda_\bullet)$ spanned by λ-objects is denoted by $\mathsf{D}_{\infty,c}(\mathcal{X}^{\mathbb{N}}_{\text{lis-et}}, \Lambda_\bullet)$

We focus on the localized ∞-category of $\mathsf{D}_{\infty,c}(\mathcal{X}^{\mathbb{N}}_{\text{lis-et}}, \Lambda_\bullet)$ by almost zero objects.

Definition. Let W be the class of those morphisms in $\mathsf{D}_{\infty,\mathrm{c}}(\mathcal{X}^{\mathbb{N}}_{\text{lis-et}}, \Lambda_\bullet)$ which are equivalences $\mathcal{M}_\bullet \to \mathcal{N}_\bullet$ with $\mathcal{N}_\bullet$ an almost zero object. We define

$$\mathbf{D}_{\infty,\mathrm{c}}(\mathcal{X}, \Lambda) := \mathsf{D}_{\infty,\mathrm{c}}(\mathcal{X}^{\mathbb{N}}_{\text{lis-et}}, \Lambda_\bullet)[W^{-1}]$$

The ∞-category $\mathbf{D}_{\infty,\mathrm{c}}(\mathcal{X}, \Lambda_\bullet)$ is stable and equipped with t-structure induced by that on $\mathsf{D}_{\infty,\mathrm{c}}(\mathcal{X}^{\mathbb{N}}_{\text{lis-et}}, \Lambda_\bullet)$. By Definition B.7.1 and Fact B.7.3 of ∞-localization, we have a functor

$$\mathsf{D}_{\infty,\mathrm{c}}(\mathcal{X}^{\mathbb{N}}_{\text{lis-et}}, \Lambda_\bullet) \longrightarrow \mathbf{D}_{\infty,\mathrm{c}}(\mathcal{X}, \Lambda).$$

On the other hand, the geometric morphism $p : \mathcal{X}^{\mathbb{N}}_{\text{lis-et}} \to \mathcal{X}_{\text{lis-et}}$ induces an adjunction

$$p_* : \mathsf{D}_{\infty,\mathrm{c}}(\mathcal{X}^{\mathbb{N}}_{\text{lis-et}}, \Lambda_\bullet) \rightleftarrows \mathsf{D}_{\infty,\mathrm{c}}(\mathcal{X}_{\text{lis-et}}, \Lambda) : p^*.$$

By Fact 7.1.1, the functor p_* factors through $\mathbf{D}_{\infty,\mathrm{c}}(\mathcal{X}^{\mathbb{N}}_{\text{lis-et}}, \Lambda_\bullet)$. The resulting functor is denoted by the same symbol as

$$p_* : \mathbf{D}_{\infty,\mathrm{c}}(\mathcal{X}, \Lambda) \longrightarrow \mathsf{D}_{\infty,\mathrm{c}}(\mathcal{X}_{\text{lis-et}}, \Lambda).$$

Definition. We define the *normalization functor* to be

$$\mathrm{Nrm} : \mathbf{D}_{\infty,\mathrm{c}}(\mathcal{X}, \Lambda) \longrightarrow \mathsf{D}_{\infty,\mathrm{c}}(\mathcal{X}^{\mathbb{N}}_{\text{lis-et}}, \Lambda_\bullet), \quad \mathcal{M} \longmapsto \mathrm{Nrm}(\mathcal{M}) := p^* p_* \mathcal{M}.$$

An object $\mathcal{M} \in \mathbf{D}_{\infty,\mathrm{c}}(\mathcal{X}, \Lambda)$ is *normalized* if the counit transformation $\mathrm{Nrm}(\mathcal{M}) \to \mathcal{M}$ is an equivalence.

Let us cite a useful criterion of normality.

Fact 7.1.2 ([LO2, 3.0.10. Proposition])**.** An object $\mathcal{M} \in \mathsf{D}_{\infty}(\mathcal{X}^{\mathbb{N}}_{\text{lis-et}}, \Lambda_\bullet)$ is normalized if and only if the morphism $\Lambda_n \otimes_{\Lambda_{n+1}} \mathcal{M}_{n+1} \to \mathcal{M}_n$ is an equivalence for every n.

By [LO2, 3.0.14. Theorem], if $\mathcal{M}$ is a λ-object, then $\mathrm{Nrm}(\mathcal{M})$ is constructible and the morphism $\mathrm{Nrm}(\mathcal{M}) \to \mathcal{M}$ has an almost zero cone. Then we have

Fact ([LO2, 3.0.18. Proposition])**.** The normalization functor sits in the adjunction

$$\mathrm{Nrm} : \mathbf{D}_{\infty,\mathrm{c}}(\mathcal{X}, \Lambda) \rightleftarrows \mathsf{D}_{\infty,\mathrm{c}}(\mathcal{X}^{\mathbb{N}}_{\text{lis-et}}, \Lambda_\bullet) : p_*.$$

Thus $\mathbf{D}_{\infty,\mathrm{c}}(\mathcal{X}, \Lambda) \in \mathsf{Cat}_\infty$.

Using the t-structure on $\mathbf{D}_{\infty,\mathrm{c}}(\mathcal{X}, \Lambda)$, let us introduce

Notation. We denote by

$$\mathbf{D}^{*}_{\infty,\mathrm{c}}(\mathcal{X}, \Lambda) \subset \mathbf{D}_{\infty,\mathrm{c}}(\mathcal{X}, \Lambda), \quad * \in \{+, -.b\}$$

the full sub-∞-category spanned by bounded objects, and by

$$\mathbf{D}^{(*)}_{\infty,\mathrm{c}}(\mathcal{X}, \Lambda) \subset \mathbf{D}_{\infty,\mathrm{c}}(\mathcal{X}, \Lambda), \quad * \in \{+, -.b\}$$

the full sub-∞-category spanned by those objects whose restriction to any quasi-compact open immersion $\mathcal{U} \hookrightarrow \mathcal{X}$ lies in $\mathbf{D}^{*}_{\infty,\mathrm{c}}(\mathcal{U}, \Lambda)$.

Finally we give a definition of the derived ∞-category of lisse-étale constructible $\overline{\mathbb{Q}_\ell}$-sheaves. We follow the 2-categorical limit method taken in [D, 1.1.3], [B, §6].

Let Λ be a discrete valuation ring with residue characteristic ℓ, Denoting by K the quotient field of Λ, we set $\mathbf{D}_{\infty,\mathrm{c}}(\mathcal{X}, K) := \mathbf{D}_{\infty,\mathrm{c}}(\mathcal{X}, \Lambda) \otimes_\Lambda K$. Running K on the ∞-category $\mathsf{FinExt}(\mathbb{Q}_\ell)$ of finite extensions of $\mathbb{Q}_\ell$, these ∞-categories form a cartesian fibration on $\mathsf{FinExt}(\mathbb{Q}_\ell)$. Thus we can take

$$\mathbf{D}_{\infty,\mathrm{c}}(\mathcal{X}, \overline{\mathbb{Q}_\ell}) := \varprojlim_{K \in \mathsf{FinExt}(\mathbb{Q}_\ell)} \mathbf{D}_{\infty,\mathrm{c}}(\mathcal{X}, K)$$

It is a stable ∞-category equipped with a t-structure.

Notation 7.1.3. We call $\mathbf{D}_{\infty,\mathrm{c}}(\mathcal{X}, \overline{\mathbb{Q}_\ell})$ the *ℓ-adic constructible derived ∞-category* of $\mathcal{X}$. An object of the heart is called an *ℓ-adic constructible sheaf* on $\mathcal{X}$.

7.2. **Internal hom and tensor functors on $\mathbf{D}_{\infty,c}$.** In the remaining part of this section, we fix a geometric derived stack $\mathcal{X}$ locally of finite presentation, and denote

$$\mathsf{D}_{\infty,c}(\mathcal{X}) := \mathsf{D}_{\infty,c}(\mathcal{X}_{\text{lis-et}}.\Lambda) \text{ or } \mathsf{D}_{\infty,c}(\mathcal{X}_{\text{lis-et}}, \overline{\mathbb{Q}_\ell}), \quad \mathbf{D}_{\infty,c}(\mathcal{X}) := \mathbf{D}_{\infty,c}(\mathcal{X}.\Lambda) \text{ or } \mathbf{D}_{\infty,c}(\mathcal{X}, \overline{\mathbb{Q}_\ell}).$$

We also use $\mathsf{D}_\infty(\mathcal{X}) := \mathsf{D}_\infty(\mathcal{X}_{\text{lis-et}}, \Lambda)$ or $\mathsf{D}_\infty(\mathcal{X}_{\text{lis-et}}, \overline{\mathbb{Q}_\ell})$. Recall also the normalization functor $\mathrm{Nrm} : \mathbf{D}_{\infty,c}(\mathcal{X}, \Lambda) \to \mathsf{D}_{\infty,c}(\mathcal{X}^{\mathbb{N}}_{\text{lis-et}}, \Lambda_\bullet)$.

We define a bifunctor $\boldsymbol{\mathscr{H}om}_\Lambda : \mathbf{D}_{\infty,c}(\mathcal{X})^{\text{op}} \times \mathbf{D}_{\infty,c}(\mathcal{X}) \to \mathsf{D}_{\infty,c}(\mathcal{X}^{\mathbb{N}}_{\text{lis-et}}, \Lambda_\bullet)$ by

$$\boldsymbol{\mathscr{H}om}_\Lambda(\mathcal{M}, \mathcal{N}) := \mathscr{H}om_{\Lambda_\bullet}(\mathrm{Nrm}\,\mathcal{M}, \mathrm{Nrm}\,\mathcal{N}).$$

Then we have

Fact 7.2.1 ([LO2, 4.0.8. Proposition])**.** $\boldsymbol{\mathscr{H}om}$ gives a bifunctor

$$\boldsymbol{\mathscr{H}om}_\Lambda : \mathbf{D}^{(-)}_{\infty,c}(\mathcal{X})^{\text{op}} \times \mathbf{D}^{(+)}_{\infty,c}(\mathcal{X}) \longrightarrow \mathbf{D}^{(+)}_{\infty,c}(\mathcal{X}).$$

Next we discuss the tensor functor. For $\mathcal{M}, \mathcal{N} \in \mathbf{D}_{\infty,c}(\mathcal{X})$, we set

$$\mathcal{M} \otimes_\Lambda \mathcal{N} := \mathrm{Nrm}(\mathcal{M}) \otimes_{\Lambda_\bullet} \mathrm{Nrm}(\mathcal{N}).$$

Thus we have a bifunctor $\otimes_\Lambda : \mathbf{D}_{\infty,c}(\mathcal{X}) \times \mathbf{D}_{\infty,c}(\mathcal{X}) \to \mathsf{D}_{\text{cart}}(\mathcal{X})$. We then have

Lemma 7.2.2 ([LO2, 6.0.12. Proposition])**.** For $\mathcal{L}, \mathcal{M}, \mathcal{N} \in \mathbf{D}^{(-)}_{\infty,c}(\mathcal{X})$ we have

$$\boldsymbol{\mathscr{H}om}_\Lambda(\mathcal{L} \otimes_\Lambda \mathcal{M}, \mathcal{N}) \simeq \boldsymbol{\mathscr{H}om}_\Lambda(\mathcal{L}, \boldsymbol{\mathscr{H}om}_\Lambda(\mathcal{M}, \mathcal{N}))$$

Proof. Denoting $\widehat{\mathcal{L}} := \mathrm{Nrm}(\mathcal{L})$ and similarly for $\mathcal{M}, \mathcal{N}$, the usual adjunction yields $\mathscr{H}om_\Lambda(\widehat{\mathcal{L}} \otimes_\Lambda \widehat{\mathcal{M}}, \widehat{\mathcal{N}}) \simeq \mathscr{H}om_\Lambda(\widehat{\mathcal{L}}, \mathscr{H}om(\widehat{\mathcal{M}}, \widehat{\mathcal{N}}))$. Using Fact 7.1.2, we can show that $\widehat{\mathcal{L}} \otimes_\Lambda \widehat{\mathcal{M}}$ is normalized. Thus we have the consequence. □

7.3. **Dualizing object.** Here we explain the dualizing object with adic coefficients following [LO2, 7].

Recall that Λ denotes a complete discrete valuation ring and $\Lambda_n = \Lambda/\mathfrak{m}^n$. Let S be an affine excellent finite-dimensional scheme where the residue characteristic ℓ of Λ is invertible and any S-schemes $f : U \to S$ of finite type has *finite cohomological dimension*. It means that there is an integer $d \in \mathbb{N}$ such that for any abelian torsion étale sheaf $\mathcal{F}$ over U we have $R^i f_* \mathcal{F} = 0$ for $i > d$. By [LO2, 7.1.3], there exists a family $\{\Omega_{S,n}, \iota_n\}_{n \in \mathbb{Z}_{>0}}$ with $\Omega_{S,n}$ a Λ_n-dualizing complex on S and $\iota : \Lambda_n \otimes_{\Lambda_n} \Omega_{S,n+1} \to \Omega_{S,n}$ an isomorphism in the bounded derived category $\mathrm{D}^b_c(S, \Lambda_n)$ of complexes of Λ-modules with constructible cohomology groups. We call it a *compatible family of dualizing complexes*.

Let k be an algebraic closure of the finite field $\mathbb{F}_q$ of order q such that ℓ is invertible. Then by [LO1, 1.0.1] the affine scheme $S = \operatorname{Spec} k$ satisfies the above conditions, and we have a compatible family $\{\Omega_{\operatorname{Spec} k,n}, \iota_n\}_{n \in \mathbb{Z}_{>0}}$ of dualizing complexes. Then, for a geometric derived stack $\mathcal{X}$ locally of finite presentation over k, we have the dualizing object

$$\Omega_{\mathcal{X},n} \in \mathsf{D}(\mathcal{X}_{\text{lis-et}}, \Lambda_n)$$

using $\Omega_{\operatorname{Spec} k,n}$ and the construction of §5.5. The isomorphism ι_n induces an equivalence

$$(\Omega_{\mathcal{X},n+1} \otimes_{\Lambda_{n+1}} \Lambda_n)_{\mathcal{U}_{\text{et}}} \longrightarrow (\Omega_{\mathcal{X},n})_{\mathcal{U}_{\text{et}}}$$

for any $\mathcal{U} \in \mathsf{dAS}^{\text{lis}}_{\mathcal{X},\text{fp}}$. By the gluing lemma (Fact 3.7.9), we have an equivalence

$$\Omega_{\mathcal{X},n+1} \otimes^{\mathrm{L}}_{\Lambda_{n+1}} \Lambda_n \xrightarrow{\sim} \Omega_{\mathcal{X}.n}.$$

In order to construct a dualizing object in $\mathbf{D}_{\infty,c}(\mathcal{X})$, let us construct a data of dualizing objects in $\mathsf{D}_{\infty,c}(\mathcal{X}^{\mathbb{N}}_{\text{lis-et}})$. Note first that the $\mathbb{N}$-simplicial ∞-topos $\mathcal{X}^{\mathbb{N}}_{\text{lis-et}}$ is equivalent to the ∞-topos associated to the following ∞-site: The underlying ∞-category of the nerve of the category whose objects are pairs $(\mathcal{U}, m) \in \mathsf{dAS}^{\text{lis}}_{\mathcal{X}} \times \mathbb{N}$, and whose set of morphisms from $(\mathcal{U}, m)$ to $(\mathcal{V}, n)$ is empty if $n > m$, and is equal to $\mathrm{Hom}_{\mathrm{h}\,\mathsf{dAS}_{\mathcal{X}}}(\mathcal{U}, \mathcal{V})$. A covering sieve is a collection $\{(\mathcal{U}_i, m_i) \to (\mathcal{U}, m)\}_{i \in I}$ with $m_i = m$ for all i and $\{\mathcal{U}_i \to \mathcal{U}\}_{i \in I} \in \mathrm{Cov}_{\text{lis-et}}(\mathcal{U})$. Then, for each $\mathcal{U} \in \mathsf{dAS}^{\text{lis,fp}}_{\mathcal{X}}$ and $m \in \mathbb{N}$, we have a sequence

$$\mathcal{X}^{\mathbb{N}}_{\text{lis-et}}\big|_{(\mathcal{U},m)} \xrightarrow{p_m} \mathcal{X}_{\text{lis-et}}|_{\mathcal{U}} \xrightarrow{\varepsilon} \mathcal{U}_{\text{et}}$$

in RTop. Here p_m is the geometric morphism defined by $p_m^{-1}(\mathcal{F}) = (\mathcal{F})_{n \le m}$. Now we set

$$\Omega_{\mathcal{U},m} := p_m^*((\Omega_{\mathcal{X},m})_{\mathcal{U}}) \simeq (\varepsilon \circ p_m)^* K_{\mathcal{U},m}\langle -d \rangle,$$

where $K_{\mathcal{U},m} \in \mathsf{D}_{\infty,c}(\mathcal{U}_{\text{et}}, \Lambda_m)$ is the dualizing object for the derived algebraic space $\mathcal{U}$, and d is the relative dimension of the smooth morphism $\mathcal{U} \to \mathcal{X}$. Then by the argument of [LO2, 7.2.3. Theorem] we can apply again the gluing lemma (Fact 3.7.9) to $\Omega_{\mathcal{U},m}$'s.

Fact ([LO2, 7.2.3. Theorem])**.** There exists a normalized object $\Omega_{\mathcal{X},\bullet} \in \mathsf{D}_{\infty,\mathrm{c}}(\mathcal{X}^{\mathbb{N}}_{\text{lis-et}})$ inducing $\Omega_{\mathcal{X},n}$, and it is unique up to contractible ambiguity.

Notation. We denote by

$$\Omega_{\mathcal{X}} \in \mathbf{D}_{\infty,\mathrm{c}}(\mathcal{X},\Lambda)$$

the image of $\Omega_{\mathcal{X},\bullet}$ under the projection $p_* : \mathsf{D}_{\infty,\mathrm{c}}(\mathcal{X}^{\mathbb{N}}_{\text{lis-et}}) \to \mathbf{D}_{\infty,\mathrm{c}}(\mathcal{X},\Lambda)$, and call it the *dualizing object*.

$\Omega_{\mathcal{X}}$ is of locally finite quasi-injective dimension. Namely, for each quasi-compact open immersion $\mathcal{U} \to \mathcal{X}$, $(\Omega_{\mathcal{X}.n})_{\mathcal{U}}$ is of finite quasi-injective dimension, and the bound depends only on $\mathcal{X}$ and Λ, not on n.

Now we define the *dualizing functor* $\mathrm{D}_{\mathcal{X}}$ by

$$\mathrm{D}_{\mathcal{X}}(\mathcal{M}) := \mathscr{H}\!\mathit{om}_{\Lambda}(\mathcal{M},\Omega_{\mathcal{X}})$$

for $\mathcal{M} \in \mathbf{D}_{\infty,\mathrm{c}}(\mathcal{X})^{\mathrm{op}}$. By Fact 7.2.1, the image under $\mathrm{D}_{\mathcal{X}}$ lies in $\mathsf{D}_{\infty,\mathrm{c}}(\mathcal{X}^{\mathbb{N}}_{\text{lis-et}},\Lambda_\bullet)$. Thus we have the induced functor

$$\mathrm{D}_{\mathcal{X}} : \mathbf{D}_{\infty,\mathrm{c}}(\mathcal{X})^{\mathrm{op}} \longrightarrow \mathbf{D}_{\infty,\mathrm{c}}(\mathcal{X}).$$

It is involutive: $\mathrm{D}_{\mathcal{X}}^2 \simeq \mathrm{id}$.

Fact ([LO2, 7.3.1. Theorem])**.** The dualizing functor $\mathrm{D}_{\mathcal{X}}$ restricts to

$$\mathrm{D}_{\mathcal{X}} : \mathbf{D}^{(-)}_{\infty,\mathrm{c}}(\mathcal{X}) \longrightarrow \mathbf{D}^{(+)}_{\infty,\mathrm{c}}(\mathcal{X})$$

We record a corollary of this fact for later use.

Lemma 7.3.1 ([LO2, 7.3.2. Corollary])**.** For $\mathcal{M},\mathcal{N} \in \mathbf{D}_{\infty,\mathrm{c}}(\mathcal{X},\Lambda)$, we have an equivalence

$$\mathscr{H}\!\mathit{om}_{\Lambda}(\mathcal{M},\mathcal{N}) \simeq \mathscr{H}\!\mathit{om}_{\Lambda}\left(\mathrm{D}_{\mathcal{X}}(\mathcal{M}),\mathrm{D}_{\mathcal{X}}(\mathcal{N})\right)$$

which is unique up to contractible ambiguity.

7.4. **Direct and inverse image functors.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of finite presentation between derived stacks locally of finite presentation. We have the induced geometric morphism $f_\bullet : \mathcal{X}^{\mathbb{N}}_{\text{lis-et}} \to \mathcal{Y}^{\mathbb{N}}_{\text{lis-et}}$, and an adjunction

$$f^* : \mathsf{D}_{\infty}(\mathcal{X}^{\mathbb{N}}_{\text{lis-et}},\Lambda_\bullet) \rightleftarrows \mathsf{D}_{\infty}(\mathcal{Y}^{\mathbb{N}}_{\text{lis-et}},\Lambda_\bullet) : f^*$$

of the derived direct image and inverse image functors.

By [LO2, 8.0.4. Proposition], if $\mathcal{M} \in \mathsf{D}_{\infty}(\mathcal{X}^{\mathbb{N}}_{\text{lis-et}},\Lambda_\bullet)$ is a left bounded λ-object, then $f_*\mathcal{M}$ is a λ-object. We can also check that AR-null objects are mapped to AR-null objects. Thus the following definition makes sense.

Notation. The obtained functor

$$f_* : \mathbf{D}^{(+)}_{\infty,\mathrm{c}}(\mathcal{X},\Lambda) \longrightarrow \mathbf{D}^{(+)}_{\infty,\mathrm{c}}(\mathcal{Y},\Lambda)$$

is called the *derived direct image functor*.

On the other hand, one can check by definition that f^* sends λ-objects to λ-objects and AR-null objects to AR-null objects. Thus we have

Notation. The obtained functor

$$f^* : \mathbf{D}_{\infty,\mathrm{c}}(\mathcal{Y},\Lambda) \longrightarrow \mathbf{D}_{\infty,\mathrm{c}}(\mathcal{X},\Lambda)$$

is called the *derived inverse image functor*.

One can check the following by the standard argument using adjunction.

Lemma 7.4.1. For $\mathcal{M} \in \mathbf{D}^{(-)}_{\infty,\mathrm{c}}(\mathcal{Y},\Lambda)$ and $\mathcal{N} \in \mathbf{D}^{(+)}_{\infty,\mathrm{c}}(\mathcal{X},\Lambda)$, we have an equivalence

$$f_*\,\mathscr{H}\!\mathit{om}_{\Lambda}(f^*\mathcal{M},\mathcal{N}) \simeq \mathscr{H}\!\mathit{om}_{\Lambda}(\mathcal{M},f_*\mathcal{N})$$

which is unique up to contractible ambiguity.

7.5. **Shriek direct and inverse image functors.** As in the finite coefficient case, we define shriek functors using dualizing functors.

Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of finite presentation between derived stacks locally of finite presentation. Let $\Omega_{\mathcal{X}}$ be the dualizing object of $\mathcal{X}$, and

$$\mathrm{D}_{\mathcal{X}} := \mathscr{H}\!\mathit{om}_{\Lambda}(-, \Omega_{\mathcal{X}}) : \mathbf{D}_{\infty,c}(\mathcal{X}, \Lambda)^{\mathrm{op}} \longrightarrow \mathbf{D}_{\infty,c}(\mathcal{Y}, \Lambda)$$

be the dualizing functor. Similarly we denote $\mathrm{D}_{\mathcal{Y}} := \mathscr{H}\!\mathit{om}_{\Lambda}(-, \Omega_{\mathcal{Y}})$.

Notation. We define the functor $f_!$ by

$$f_! := \mathrm{D}_{\mathcal{Y}} \circ f_* \circ \mathrm{D}_{\mathcal{X}} : \mathbf{D}^{(-)}_{\infty,c}(\mathcal{X}, \Lambda) \longrightarrow \mathbf{D}^{(-)}_{\infty,c}(\mathcal{Y}, \Lambda),$$

and the functor $f^!$ by

$$f^! := \mathrm{D}_{\mathcal{Y}} \circ f^* \circ \mathrm{D}_{\mathcal{X}} : \mathbf{D}_{\infty,c}(\mathcal{Y}, \Lambda) \longrightarrow \mathbf{D}_{\infty,c}(\mathcal{X}, \Lambda).$$

By Lemma 7.4.1 and Lemma 7.3.1, we immediately have

Lemma. For $\mathcal{M} \in \mathbf{D}^{(-)}_{\infty,c}(\mathcal{X}, \Lambda)$ and $\mathcal{N} \in \mathbf{D}^{(+)}_{\infty,c}(\mathcal{Y}, \Lambda)$, we have an equivalence

$$f_* \mathscr{H}\!\mathit{om}_{\Lambda}(\mathcal{M}, f^! \mathcal{N}) \simeq \mathscr{H}\!\mathit{om}_{\Lambda}(f_! \mathcal{M}, \mathcal{N})$$

which is unique up to contractible ambiguity.

We also have

Lemma ([LO2, 9.1.2. Lemma])**.** If f is smooth of relative dimension d, then for any $\mathcal{M} \in \mathbf{D}_{\infty,c}(\mathcal{X}, \Lambda)$ we have an equivalence

$$f^! \mathcal{M} \simeq f^* \mathcal{M}\langle d \rangle$$

which is unique up to contractible ambiguity.

Proof. We have $\Omega_{\mathcal{X}} \simeq f^* \Omega_{\mathcal{Y}}\langle d \rangle$ by definition of the dualizing object and Lemma 6.6.4 (1). Then by the biduality $\mathrm{D}^2_{\mathcal{X}} \simeq \mathrm{id}$ we have the consequence. □

Finally we explain the smooth base change with adic coefficients. Let

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{p'} & \mathcal{X} \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{Y}' & \xrightarrow[p]{} & \mathcal{Y} \end{array}$$

be a cartesian square of derived stacks with f of finite type. Then we have a morphism

$$\alpha : (p')^* f_! \longrightarrow f'_! (p')^*$$

of functors $\mathbf{D}^{(+)}_{\infty,c}(\mathcal{X}, \Lambda) \to \mathbf{D}^{(-)}_{\infty,c}(\mathcal{Y}', \Lambda)$. By Proposition 6.6.5 we have

Proposition 7.5.1. If p is smooth, then α is an equivalence.

We close this section by

Remark 7.5.2. (1) All the claims hold for the ℓ-adic constructible derived ∞-category $\mathbf{D}_{\infty,c}(\mathcal{X}, \overline{\mathbb{Q}_\ell})$ (Notation 7.1.3).
(2) If we take $\mathcal{X}$ to be an algebraic stack of finite presentation, then we can recover the categories and functors in [LO2].

8. Perverse sheaves on derived stacks

In this section we introduce the perverse t-structure on the constructible derived ∞-category on a derived stack $\mathcal{X}$, and discuss perverse sheaves, the decomposition theorem and weights. Our argument basically follows [LO3], where the theory of perverse sheaves on an algebraic stack is developed.

Let k be a fixed field, and Λ be a complete discrete valuation ring whose residue characteristic is invertible in k. We denote $\Lambda_n := \Lambda/\mathfrak{m}^{n+1}$ for $n \in \mathbb{N}$. We fix a geometric derived stack $\mathcal{X}$ locally of finite presentation over k.

8.1. **Gluing of t-structures.** We recollect standard facts on gluing of t-structures in [BBD], specializing to the constructible derived ∞-categories of derived stacks. See also [LO3, §2].

Assumption 8.1.1. Let D, D_U and D_F be stable ∞-categories, and

$$\mathsf{D}_F \xrightarrow{i_*} \mathsf{D} \xrightarrow{j^*} \mathsf{D}_U$$

be a sequence of exact functors (Definition D.1.6). Assume the following conditions hold.

(i) i_* has a left adjoint i^* and a right adjoint $i^!$.
(ii) j^* has a left adjoint $j_!$ and a right adjoint j_*.
(iii) $i^! j_* = 0$.
(iv) For each $K \in \mathsf{D}$, there exist morphisms $i_* i^* K \to j_! j^* K[1]$ and $j_* j^* K \to i_* i^! K[1]$ in D such that the induced triangles

$$j_! j^* K \to K \to i_* i^* K \to j_! j^* K[1], \quad i_* i^! K \to K \to j_* j^* K \to i_* i^! K[1]$$

in $\mathrm{h}\,\mathsf{D}$ is distinguished.
(v) All the unit and counit transformations $i^* i_* \to \mathrm{id} \to i^! i_*$ and $j^* j_* \to \mathrm{id} \to j^* j_!$ are equivalences.

Fact 8.1.2. Under Assumption 8.1.1, we further suppose that D_F and D_U are equipped with t-structures determined by $(\mathsf{D}_F^{\le 0}, \mathsf{D}_F^{\ge 0})$ and $(\mathsf{D}_U^{\le 0}, \mathsf{D}_U^{\ge 0})$ respectively. Define the full sub-∞-categories $\mathsf{D}^{\le 0}, \mathsf{D}^{\ge 0} \subset \mathsf{D}$ by

$$\mathsf{D}^{\le 0} := \{K \in \mathsf{D} \mid j^* K \in \mathsf{D}_U^{\le 0}, i^* K \in \mathsf{D}_F^{\le 0}\}, \quad \mathsf{D}^{\ge 0} := \{K \in \mathsf{D} \mid j^* K \in \mathsf{D}_U^{\ge 0}, i^! K \in \mathsf{D}_F^{\ge 0}\}.$$

Then the pair $(\mathsf{D}^{\le 0}, \mathsf{D}^{\ge 0})$ determines a t-structure on D.

We consider two cases which satisfy Assumption 8.1.1.

Let k, Λ and $\mathcal{X}$ be as in the beginning of this section. Let $i : \mathcal{F} \to \mathcal{X}$ be a closed immersion of a derived stack and $\iota : \mathcal{U} \to \mathcal{X}$ be the open immersion of its complement (Notation 2.2.25). Recall that we also have constructed derived functors $i_! : \mathsf{D}_{\mathcal{F}} \to \mathsf{D}$ and $j^! : \mathsf{D} \to \mathsf{D}_{\mathcal{U}}$, By §6.6 we have $i_! = i_*$ and $j^! \simeq j^*$, so that they are compatible with the notation in Assumption 8.1.1. Then, by the argument in §6.6, we have

Lemma. Fix $n \in \mathbb{N}$. Then the stable ∞-categories

$$\mathsf{D} := \mathsf{D}^b_{\infty,\mathrm{c}}(\mathcal{X}_{\text{lis-et}}, \Lambda_n), \quad \mathsf{D}_{\mathcal{F}} := \mathsf{D}^b_{\infty,\mathrm{c}}(\mathcal{F}_{\text{lis-et}}, \Lambda_n), \quad \mathsf{D}_{\mathcal{U}} := \mathsf{D}^b_{\infty,\mathrm{c}}(\mathcal{U}_{\text{lis-et}}, \Lambda_n),$$

and the direct and inverse image functors

$$\mathsf{D}_{\mathcal{F}} \xrightarrow{i_*} \mathsf{D} \xrightarrow{j^*} \mathsf{D}_{\mathcal{U}}$$

constructed in §6 satisfy Assumption 8.1.1.

We can also consider the adic coefficient case.

Lemma. The stable ∞-categories

$$\mathbf{D} := \mathbf{D}^b_{\infty,\mathrm{c}}(\mathcal{X}_{\text{lis-et}}, \Lambda), \quad \mathbf{D}_{\mathcal{F}} := \mathbf{D}^b_{\infty,\mathrm{c}}(\mathcal{F}_{\text{lis-et}}, \Lambda), \quad \mathbf{D}_{\mathcal{U}} := \mathbf{D}^b_{\infty,\mathrm{c}}(\mathcal{U}_{\text{lis-et}}, \Lambda),$$

and the functors

$$\mathbf{D}_{\mathcal{F}} \xrightarrow{i_*} \mathbf{D} \xrightarrow{j^*} \mathbf{D}_{\mathcal{U}}$$

constructed in §6 satisfy Assumption 8.1.1.

8.2. **Perverse t-structure.**

8.2.1. *The case of derived algebraic spaces.* Here we will introduce the perverse t-structure for derived algebraic spaces. Let k and $\Lambda = \varprojlim_n \Lambda_n$ be as in the beginning of this section.

Consider a derived algebraic space $\mathcal{U}$ of finite presentation over k. Fix $n \in \mathbb{N}$ for a while, and let $\mathsf{D}^b := \mathsf{D}^b_{\mathrm{c}}(\mathcal{U}, \Lambda_n)$ be the derived ∞-category of bounded complexes of étale Λ_n-sheaves with constructible cohomology groups.

Following the case of algebraic spaces (Definition A.1.7), let us introduce

Definition. A *point* of a derived algebraic space $\mathcal{U}$ is a monomorphism $\operatorname{dSpec} L \to \mathcal{U}$ of derived stacks (Definition 2.2.11) with L some field. It will be denoted typically as $i_u : u \to \mathcal{U}$.

The equivalence of two points on $\mathcal{U}$ is defined similarly as Definition A.1.7. The dimension of $\mathcal{U}$ at its point u is also defined in the standard manner, and it will be denoted by $\dim(u)$.

We denote by

$$ {}^{\mathrm{p}}\,\mathsf{D}^{b,\le 0} \subset \mathsf{D}^b $$

the full sub-∞-category of objects K such that $H^j(i_u^* K) = 0$ for each point u of $\mathcal{U}$ and every $j > -\dim(u)$. Here $i_u^* K := (i_{\overline{u}}^* K)_u$ with $i_{\overline{u}} : \overline{u}_{\mathrm{red}} \to \mathcal{U}$ the closed immersion. Note that the symbol $\overline{X}_{\mathrm{red}}$ makes sense here. Similarly, we denote by

$$ {}^{\mathrm{p}}\,\mathsf{D}^{b,\ge 0} \subset \mathsf{D}^b $$

the sub-∞-category of those K such that $H^j(i_u^! K) = 0$ for each point $u \in \mathcal{U}$ and every $j < -\dim(u)$. Then, as in the case of schemes [BBD, 2.2.11], the pair

$$ ({}^{\mathrm{p}}\,\mathsf{D}^{b,\le 0}, {}^{\mathrm{p}}\,\mathsf{D}^{b,\ge 0}) $$

determines a t-structure on D^b, which is called the (middle) *perverse t-structure.* We denote by

$$ {}^{\mathrm{p}}H^0 : \mathsf{D}^b \longrightarrow \mathsf{D}^b $$

the perverse cohomology functor.

In [LO3, §3], an extension is developed of the perverse t-structure on the bounded derived category to the unbounded derived category. Let us explain it in the context of derived algebraic spaces. For $K \in \mathsf{D} = \mathsf{D}_{\mathrm{c}}(\mathcal{U}, \Lambda_n)$ and $\alpha, \beta \in \mathbb{Z}$ with $\alpha \le \beta$, we denote $\tau_{[\alpha,\beta]} K := \tau_{\ge\alpha}\tau_{\le\beta} K$. Then we have

Fact ([LO3, Lemma 3.3])**.** For any $K \in \mathsf{D}^b$, there exist $\alpha, \beta \in \mathbb{Z}$ such that $\alpha \le \beta$ and ${}^{\mathrm{p}}H^0(K) \simeq {}^{\mathrm{p}}H^0(\tau_{\alpha,\beta} K)$.

For the completeness of presentation, let us explain the outline of the proof. It suffices to show that there exist $\alpha, \beta \in \mathbb{Z}$, $\alpha < \beta$ such that for any $K \in \mathsf{D}^{<\alpha}$ or $K \in \mathsf{D}^{>\beta}$ we have ${}^{\mathrm{p}}H^0(K) = 0$. By the definition of the perverse sheaf, one can take α to be an integer smaller then $-\dim X$. Since the dualizing sheaf of a scheme of finite type over k has finite quasi-injective dimension, there exists $c \in \mathbb{N}$ such that for any $d \in \mathbb{Z}$, any point u of $\mathcal{U}$ and $K \in \mathsf{D}^{>d}$ we have $i_u^! K \in \mathsf{D}^{>d+c}$. Thus we can take β to be an integer greater than $-c$.

Using the integers $\alpha < \beta$ in the above Fact, we have a well-defined functor

$$ {}^{\mathrm{p}}H : \mathsf{D} \longrightarrow \mathsf{D}^b, \quad K \longmapsto {}^{\mathrm{p}}H(\tau_{[\alpha,\beta]} K). $$

We now define ${}^{\mathrm{p}}\,\mathsf{D}^{\le 0}$ (resp. ${}^{\mathrm{p}}\,\mathsf{D}^{\ge 0}$) to be the full sub-$\infty$-category of D spanned by those $K \in \mathsf{D}$ with ${}^{\mathrm{p}}\pi_j(K) := {}^{\mathrm{p}}H(K[j]) = 0$ for any $j \in \mathbb{Z}_{\le 0}$ (resp. $j \in \mathbb{Z}_{\ge 0}$). The pair

$$ ({}^{\mathrm{p}}\mathsf{D}^{\le 0}, {}^{\mathrm{p}}\mathsf{D}^{\ge 0}) $$

determines a t-structure on D, which is called the *perverse t-structure.*

Similarly, for the derived ∞-category $\mathbf{D}^b := \mathbf{D}_{\mathrm{c}}^b(\mathcal{U}, \Lambda)$ with adic coefficients, we have a pair

$$ ({}^{\mathrm{p}}\,\mathbf{D}^{b,\le 0}, {}^{\mathrm{p}}\,\mathbf{D}^{b,\ge 0}) $$

given by the same condition, and it determines a t-structure on $\mathbf{D}^b$ (see [LO3, Proposition 3.1] for the detail). We also call the obtained t-structure the *perverse t-structure.*

The same reasoning works for $\mathbf{D} := \mathbf{D}_{\mathrm{c}}(X, \Lambda)$.

8.2.2. *The case of derived stacks.* Now we consider a geometric derived stack $\mathcal{X}$. We have the stable ∞-categories $\mathsf{D}_{\infty,\mathrm{c}}(\mathcal{X}, \Lambda_n)$ and $\mathbf{D}_{\infty,\mathrm{c}}(\mathcal{X}, \Lambda)$. We denote either of them by $\mathsf{D}(\mathcal{X})$.

We first assume that $\mathcal{X}$ is of finite presentation. Take a smooth presentation $\mathcal{X}_\bullet \to \mathcal{X}$. We denote by $p : \mathcal{X}_0 \to \mathcal{X}$ the projection from the derived algebraic space $\mathcal{X}_0$. Then we may assume that $\mathcal{X}_0$ is of finite presentation.

Definition. For a derived stack $\mathcal{X}$ of finite presentation, we define

$$ {}^{\mathrm{p}}\mathsf{D}^{\le 0}(\mathcal{X}) \quad (\text{resp. } {}^{\mathrm{p}}\mathsf{D}^{\ge 0}(\mathcal{X})) $$

to be the full sub-∞-category of $\mathsf{D}(\mathcal{X})$ spanned by those objects $\mathcal{M}$ such that $p^*\mathcal{M}[d] \in {}^{\mathrm{p}}\mathsf{D}^{\le 0}(\mathcal{X}_0)$ (resp. $p^*\mathcal{M}[d] \in {}^{\mathrm{p}}\mathsf{D}^{\ge 0}(\mathcal{X}_0)$), where d is the relative dimension of $p : \mathcal{X}_0 \to \mathcal{X}$.

Lemma. The ∞-categories ${}^{\mathrm{p}}\mathsf{D}^{\le 0}(\mathcal{X})$ and ${}^{\mathrm{p}}\mathsf{D}^{\ge 0}(\mathcal{X})$ are independent of the choice of $\mathcal{X}_\bullet \to \mathcal{X}$ up to equivalence.

Proof. The proof of [LO3, Lemma 4.1] works. □

Lemma 8.2.1. For a geometric derived stack $\mathcal{X}$ of finite presentation, the pair $({}^{\mathrm{p}}\mathsf{D}^{\le 0}(\mathcal{X}), {}^{\mathrm{p}}\mathsf{D}^{\ge 0}(\mathcal{X}))$ determines a t-structure on $\mathsf{D}(\mathcal{X})$.

Proof. The non-trivial point is to check the condition (iii) of the t-structure (Definition D.2.1). Using noetherian induction and gluing of t-structures (Fact 8.1.2), we can construct for each $\mathcal{M} \in \mathsf{D}(\mathcal{X})$ a fiber sequence $\mathcal{M}' \to \mathcal{M} \to \mathcal{M}''$ with $\mathcal{M}' \in \mathsf{D}^{\geq 0}(\mathcal{X})$ and $\mathcal{M}'' \in \mathsf{D}^{\leq 0}(\mathcal{X})$. See [LO3, Proposition 3.1] □

Next we assume that $\mathcal{X}$ is locally of finite presentation. In this case, we have

Proposition 8.2.2. For a geometric derived stack $\mathcal{X}$ locally of finite presentation, we define

$$^{\mathrm{p}}\mathsf{D}^{\leq 0}(\mathcal{X}) \quad (\text{resp. } ^{\mathrm{p}}\mathsf{D}^{\geq 0}(\mathcal{X}))$$

to be the full sub-∞-category of $\mathsf{D}(\mathcal{X})$ spanned by those objects $\mathcal{M}$ such that for each open immersion $\mathcal{Y} \to \mathcal{X}$ with $\mathcal{Y}$ a derived stack of of finite presentation, the restriction $\mathcal{M}|_{\mathcal{Y}}$ belongs to $^{\mathrm{p}}\mathsf{D}^{\leq 0}(\mathcal{Y})$ (resp. to $^{\mathrm{p}}\mathsf{D}^{\geq 0}(\mathcal{Y})$). Then the pair

$$(^{\mathrm{p}}\mathsf{D}^{\leq 0}(\mathcal{X}), {}^{\mathrm{p}}\mathsf{D}^{\geq 0}(\mathcal{X}))$$

determines a t-structure on $\mathsf{D}(\mathcal{X})$, which will be called the *perverse t-structure.*

Proof. We follow the proof of [LO3, Theorem 5.1]. The non-trivial point is the condition (iii) of the t-structure. We can write $\mathcal{X} \simeq \varinjlim_{i \in I} \mathcal{X}_i$ with $\{\mathcal{X}_i\}_{i \in I}$ a filtered family of open derived substacks of finite presentation. For each $i \in I$, the restriction $\mathcal{M}|_{\mathcal{X}_i}$ sits in a fiber sequence $\mathcal{M}'_i \to \mathcal{M}|_{\mathcal{X}_i} \to \mathcal{M}''_i$ by Lemma 8.2.1. Denote by $j_i : \mathcal{X}_i \to \mathcal{X}$ the open immersion. Then we have a sequence

$$(j_i)_! \mathcal{M}'_i \to (j_{i+1})_! \mathcal{M}'_{i+1} \to \cdots$$

in $\mathsf{D}(\mathcal{X})$, so we can define $\mathcal{M}'$ to be the colimit of this sequence. We then have a morphism $\mathcal{M}' \to \mathcal{M}$, and also a fiber sequence $\mathcal{M}' \to \mathcal{M} \to \mathcal{M}''$. By [LO3, Lemma 5.2], the restriction of this fiber sequence to $\mathcal{X}_i$ is equivalent to $\mathcal{M}'_i \to \mathcal{M}|_{\mathcal{X}_i} \to \mathcal{M}''_i$ for each $i \in I$. Thus we have $\mathcal{M}' \in {}^{\mathrm{p}}\mathsf{D}^{\leq 0}(\mathcal{X})$ and $\mathcal{M}' \in {}^{\mathrm{p}}\mathsf{D}^{\geq 0}(\mathcal{X})$. □

Definition 8.2.3. For a derived stack $\mathcal{X}$ locally of finite presentation, we define the *perverse t-structure* of $\mathsf{D}(\mathcal{X}) = \mathsf{D}_{\infty,\mathrm{c}}(\mathcal{X}, \Lambda_n)$ or $\mathbf{D}_{\infty,\mathrm{c}}(\mathcal{X}, \Lambda)$ to be the t-structure given by Proposition 8.2.2. Its heart is denoted by

$$\mathsf{Perv}(\mathcal{X}) \subset \mathsf{D}(\mathcal{X}),$$

and its object is called a *perverse sheaf* on $\mathcal{X}$.

We denote by

$$^{\mathrm{p}}H^0 : \mathsf{D}(\mathcal{X}) \longrightarrow \mathsf{Perv}(\mathcal{X})$$

the perverse cohomology functor. The standard argument in [BBD] gives

Lemma. The homotopy category $\mathrm{h}\,\mathsf{Perv}(\mathcal{X})$ is abelian which is artinian and noetherian.

In particular, we have the notion of *simple objects* in $\mathsf{Perv}(\mathcal{X})$. They are called *simple perverse sheaves* on $\mathcal{X}$.

Now let us introduce intermediate extensions. Recall that for an open immersion j of derived stacks we have a morphism $j_! \to j_*$ of derived functors (Lemma 6.6.4).

Definition. Let $i : \mathcal{F} \to \mathcal{X}$ be a closed substack with the complement $j : \mathcal{U} \to \mathcal{X}$. For a perverse sheaf $\mathcal{P}$ on $\mathcal{U}$, we define $j_{!*}\mathcal{P} \in \mathsf{Perv}(\mathcal{X})$ to be the image in the abelian category $\mathrm{h}\,\mathsf{Perv}(\mathcal{X})$ of the morphism $^{\mathrm{p}}\pi_0(j_!\mathcal{P}) \to {}^{\mathrm{p}}\pi_0(j_*\mathcal{P})$:

$$j_{!*}\mathcal{P} := \mathrm{Im}(^{\mathrm{p}}\pi_0(j_!\mathcal{P}) \to {}^{\mathrm{p}}\pi_0(j_*\mathcal{P})).$$

We call it the *intermediate extension.*

Note that as a perverse sheaf, or an object of $\mathsf{Perv}(\mathcal{X})$, $j_{!*}\mathcal{P}$ is defined up to contractible ambiguity. Hereafter we consider $j_{!*}$ as the functor

$$j_{!*} : \mathsf{Perv}(\mathcal{U}) \longrightarrow \mathsf{Perv}(\mathcal{X}).$$

Recall the dualizing functor $\mathrm{D}_{\mathcal{X}}$ on $\mathsf{D}(\mathcal{X})$ (see §7.5, Remark 7.5.2). Here are standard properties of the intermediate extension.

Lemma. (1) We have $j^*(j_{!*}\mathcal{P}) \simeq \mathcal{P}$ and $^{\mathrm{p}}\pi_0(i^*(j_{!*}\mathcal{P})) = 0$, and these properties determine $j_{!*}\mathcal{P}$ as an object of $\mathsf{Perv}(\mathcal{X})$ uniquely up to contractible ambiguity.

(2) Let $p : \mathcal{X}_\bullet \to \mathcal{X}$ be a smooth presentation of relative dimension d, and let $\mathcal{F}_0 \xrightarrow{i'} \mathcal{X}_0 \xrightarrow{j'} \mathcal{U}_0$ be the pullbacks of $\mathcal{F}$ and $\mathcal{U}$. Then $p^*[d] j_{!*} \simeq j'_{!*} p^*[d]$

Proof. The proofs of [LO3, Lemma 6.1, 6.2] works with the case of derived algebraic spaces, the derived functors in §7 and the smooth base change (Proposition 7.5.1). □

For later use, let us introduce

Definition 8.2.4. (1) An object $\mathcal{M}_\bullet = (\mathcal{M}_n)_{n\in\mathbb{N}} \in \mathsf{Mod}_{\Lambda_\bullet}(\mathcal{X})$ is *smooth* if all $\mathcal{M}_n$ are locally constant (Definition 5.4.1).

(2) An object $\mathcal{M} \in \mathsf{D}(\mathcal{X})$ is *smooth* if for each $j \in \mathbb{Z}$ the homotopy group $\pi_j\mathcal{M}$ is represented by a smooth object of $\mathsf{D}_{\infty,\mathrm{c}}(\mathcal{X}, \Lambda_\bullet)$, and vanishes for almost all j.

8.3. **Decomposition theorem.** We now discuss the decomposition theorem of perverse sheaves on derived stacks. We fix a base field k. Let $\mathcal{X}$ be a geometric derived stack of finite presentation over k. We have the derived ∞-category $\mathbf{D}_{\infty,\mathrm{c}}(\mathcal{X}, \mathbb{Z}_\ell)$ of constructible $\mathbb{Z}_\ell$-sheaves with ℓ invertible in k.

Let us consider the full sub-∞-category $\mathbf{D}^b_{\infty,\mathrm{c}}(\mathcal{X}, \mathbb{Z}_\ell)$ spanned by bounded objects, and set $\mathbf{D}^b_{\infty,\mathrm{c}}(\mathcal{X}, \mathbb{Q}_\ell) := \mathbf{D}^b_{\infty,\mathrm{c}}(\mathcal{X}, \mathbb{Z}_\ell) \otimes \mathbb{Q}_\ell$. By the argument in the previous subsection, we have the perverse t-structure on $\mathbf{D}^b_{\infty,\mathrm{c}}(\mathcal{X}, \mathbb{Z}_\ell)$. It induces a t-structure on $\mathbf{D}^b_{\infty,\mathrm{c}}(\mathcal{X}, \mathbb{Q}_\ell)$, which will also be called the perverse t-structure.

Definition. An object in the heart of this t-structure will be called a *perverse $\mathbb{Q}_\ell$-sheaf*.

For a derived stack $\mathcal{V}$ over the base field k, we have a derived stack over the algebraic closure $\overline{k}$ with the *reduced structure*, which will be denote by $(\mathcal{V} \otimes_k \overline{k})_{\mathrm{red}}$. Let us give a local explanation on the reduced derived stack: For a derived k-algebra $A = \oplus_{n\in\mathbb{N}} A_n \in \mathsf{sCom}_k$, we denote $A_{\mathrm{red}} := \oplus_{n\in\mathbb{N}} A_{\mathrm{red},n}$ with $A_{\mathrm{red},0} := (A_0)_{\mathrm{red}}$ as a commutative k-algebra and $A_{\mathrm{red},n} := A_n \otimes_{A_0} A_{\mathrm{red},0}$ for $n > 0$.

We can now describe simple perverse $\mathbb{Q}_\ell$-sheaves on $\mathcal{X}$. Let us call a derived substack $\mathcal{V} \hookrightarrow \mathcal{X}$ *irreducible* if $\mathcal{V}$ is truncated (Definition 2.2.27) and the truncation $\mathrm{Trc}\,\mathcal{V}$ is an irreducible algebraic stack.

Proposition 8.3.1 (c.f. [LO3, Theorem 8.2])**.** Let $j : \mathcal{V} \to \mathcal{X}$ be the closed embedding of an irreducible substack such that $(\mathcal{V} \otimes_k \overline{k})_{\mathrm{red}}$ is smooth. Let $\mathcal{L} \in \mathsf{Mod}_{\mathbb{Q}_\ell}(\mathcal{V})$ be smooth (Definition 8.2.4) and a simple object (in the abelian category $\mathrm{h}\,\mathsf{Mod}_{\mathbb{Q}_\ell}(\mathcal{V})$). Then the intermediate extension $j_{!*}(\mathcal{L}[\dim(\mathcal{V})])$ is a simple perverse $\mathbb{Q}_\ell$-sheaf on $\mathcal{X}$, and every simple perverse $\mathbb{Q}_\ell$-sheaf on $\mathcal{X}$ is obtained in this way.

The argument in [LO3, §8] works in our situation with obvious modifications, and we omit the detail of the proof.

9. Moduli stack of perfect dg-modules

In this section we cite from [TVa] the theory of moduli stacks of modules over dg-categories via derived stacks.

9.1. **Dg-categories.** In this subsection we collect basic notions on dg-categories. We fix a commutative ring k, and denote by $\mathrm{C}(k)$ the category of complexes of k-modules with the standard monoidal structure $\otimes_k$. For later use, we write the dependence on the universe explicitly. See §0.2 for our convention on the universe.

Definition 9.1.1. A $\mathbb{U}$-small *dg-category* D over k consists of the following data.

- A $\mathbb{U}$-small set $\mathrm{Ob}(\mathrm{D})$, which is called the set of objects of D. We denote $X \in \mathrm{D}$ to mean that $X \in \mathrm{Ob}(\mathrm{D})$.
- For every $X, Y \in \mathrm{D}$, a complex of k-modules

$$\cdots \xrightarrow{d_{-2}} \mathrm{Hom}_{\mathrm{D}}(X,Y)^{-1} \xrightarrow{d_{-1}} \mathrm{Hom}_{\mathrm{D}}(X,Y)^{0} \xrightarrow{d_0} \mathrm{Hom}_{\mathrm{D}}(X,Y)^{1} \xrightarrow{d_1} \cdots$$

  called the complex of morphisms from X to Y, which is denoted just by $\mathrm{Hom}_{\mathrm{D}}(X,Y)$.
- For every $X, Y, Z \in \mathrm{D}$, a morphism of complexes

$$\circ : \mathrm{Hom}_{\mathrm{D}}(Y,Z) \otimes_k \mathrm{Hom}_{\mathrm{D}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathrm{D}}(X,Z)$$

  called the composition map, which is to satisfy the unit and associativity conditions.

In other words, a dg-category over k is nothing but a $\mathrm{C}(k)$-enriched category. Hereafter the word "a dg-category" means a $\mathbb{U}$-small dg-category over k.

Notation 9.1.2. We denote by k_{dg} the dg-category over k with one object $*$ and the complex of morphisms $* \to *$ given by k.

We will always consider the projective model structure on $\mathrm{C}(k)$, where a fibration is defined to be an epimorphism and a weak equivalence is defined to be a quasi-isomorphism (see [H, §2.3]).

Recall also the notion of *dg-functors*.

Definition. A *dg-functor* $f : \mathrm{D} \to \mathrm{D}'$ between dg-categories D and D′ consists of

- A map $f : \mathrm{Ob}(\mathrm{D}) \to \mathrm{Ob}(\mathrm{D}')$ of sets.
- For every $x, y \in \mathrm{D}$, a morphism $f_{x,y} : \mathrm{Hom}_{\mathrm{D}}(x, y) \to \mathrm{Hom}_{\mathrm{D}'}(f(x), f(y))$ in $\mathrm{C}(k)$ satisfying the unit and associativity conditions.

Notation 9.1.3. We denote by $\mathrm{dgCat}_{\mathbb{U}}$ the category of $\mathbb{U}$-small dg-categories and dg-functors between them.

Let us recall some standard notions on dg-functors. For that, we introduce

Notation 9.1.4. For a dg-category D, we denote by [D] the category with

$$\mathrm{Ob}([\mathrm{D}]) := \mathrm{Ob}(\mathrm{D}), \quad \mathrm{Hom}_{[\mathrm{D}]}(x, y) := H^0(\mathrm{Hom}_{\mathrm{D}}(x, y)).$$

Here H^0 denotes the 0-th cohomology of a complex.

Definition 9.1.5 ([T2, Definition 2.1])**.** Let $f : \mathrm{D} \to \mathrm{D}'$ be a dg-functor between dg-categories.

(1) f is *quasi-fully faithful* if $f_{x,y} : \mathrm{Hom}_{\mathrm{D}}(x, y) \to \mathrm{Hom}_{\mathrm{D}'}(f(x), f(y))$ is a quasi-isomorphism in $\mathrm{C}(k)$ for any $x, y \in \mathrm{D}$.
(2) f is *quasi-essentially surjective* if the induced functor $[f] : [\mathrm{D}] \to [\mathrm{D}']$ is essentially surjective.
(3) f is a *quasi-equivalence* if it is quasi-fully faithful and quasi-essentially surjective.

Finally we introduce

Notation. We denote by D^{op} the *opposite* of a dg-category D. It is a dg-category given by $\mathrm{Ob}(\mathrm{D}^{\mathrm{op}}) := \mathrm{Ob}(\mathrm{D})$, $\mathrm{Hom}_{\mathrm{D}^{\mathrm{op}}}(x, y) := \mathrm{Hom}_{\mathrm{D}}(y, x)$ and the composition map with an appropriate sign.

9.2. **Perfect dg modules.** In this subsection we recall the notion of perfect dg-modules. The main references are [T2] and [TVa, §2]. We fix a dg-category D over k.

Definition. A D-*dg-module* is a $\mathrm{C}(k)$-enriched functor $\mathrm{D} \to \mathrm{C}(k)$. We denote the category of D-dg-modules over by $\mathrm{Mod}_{\mathrm{dg}}(\mathrm{D})$.

The category $\mathrm{Mod}_{\mathrm{dg}}(\mathrm{D})$ has a model structure such that a morphism $f : F \to G$ is a weak equivalence (resp. fibration) if for any $z \in \mathrm{D}$ the morphism $f_z : F(z) \to G(z)$ is a weak equivalence (resp. fibration) in $\mathrm{C}(k)$. We always regard $\mathrm{Mod}_{\mathrm{dg}}(\mathrm{D})$ as a model category by this structure. Then

Lemma 9.2.1. The model category $\mathrm{Mod}_{\mathrm{dg}}(\mathrm{D})$ is stable in the sense of [H, §7], so that the homotopy category $\mathrm{Ho}\,\mathrm{Mod}_{\mathrm{dg}}(\mathrm{D})$ has a natural triangulated structure whose triangles are the image of of homotopy fiber sequences.

The dg structure and the model structure make $\mathrm{Mod}_{\mathrm{dg}}(\mathrm{D})$ a $\mathrm{C}(k)$-model category in the sense of [H, Definition 4.2.18]. So, let us give an interlude on $\mathrm{C}(k)$-model categories.

Notation. For a $\mathrm{C}(k)$-model category M, we denote by $\mathrm{M}^\circ \subset \mathrm{M}$ the model subcategory of fibrant-cofibrant objects (using the same symbol as §B.6). We endow M° with the dg structure by restriction of that on M, and consider M° as a $\mathrm{C}(k)$-model category.

In [TVa], M° is denoted by $\mathrm{Int}(M)$. For a $\mathrm{C}(k)$-model category M, we have an equivalence

$$\mathrm{Ho}(\mathrm{M}) \simeq [\mathrm{M}^\circ] \tag{9.2.1}$$

by [T2, Proposition 3.5]. Here $[\mathrm{M}^\circ]$ is the category arising from the dg structure (Notation 9.1.4), and Ho(M) is the homotopy category arising from the model structure.

Let us return to the discussion on $\mathrm{Mod}_{\mathrm{dg}}(\mathrm{D})$, We denote by

$$\mathrm{Mod}_{\mathrm{dg}}(\mathrm{D})^\circ \subset \mathrm{Mod}_{\mathrm{dg}}(\mathrm{D})$$

the full sub-dg-category of fibrant-cofibrant objects (using the same symbol as in §B.6). By the discussion in [TVa, §2.2 pp. 399–400], any object of $\mathrm{Mod}_{\mathrm{dg}}(\mathrm{D})$ is fibrant, so that $\mathrm{Mod}_{\mathrm{dg}}(\mathrm{D})^\circ$ is actually equivalent to the full sub-dg-category of cofibrant objects. Note that in [TVa] it is denoted by $\widehat{\mathrm{D}^{\mathrm{op}}}$.

Definition. A D-dg-module $M \in \mathrm{Mod}_{\mathrm{dg}}(\mathrm{D})^\circ$ is called *perfect* if it is homotopically finitely presented in the model category $\mathrm{Mod}_{\mathrm{dg}}(\mathrm{D})$. We denote by

$$\mathrm{P}(\mathrm{D}) := \mathrm{Mod}_{\mathrm{dg}}(\mathrm{D})^\circ_{\mathrm{pe}} \subset \mathrm{Mod}_{\mathrm{dg}}(\mathrm{D})^\circ$$

the full sub-dg-category of perfect objects.

In other words, for a perfect D-dg-module M, the natural morphism

$$\varinjlim_{i\in I} \mathrm{Map}_{\mathrm{Mod}_{\mathrm{dg}}(\mathrm{D})}(M, N_i) \longrightarrow \mathrm{Map}_{\mathrm{Mod}_{\mathrm{dg}}(\mathrm{D})}(M, \varinjlim_{i\in I} N_i)$$

is an isomorphism in the topological category $\mathcal{H}$ of spaces (Definition 1.3.2) for any filtered system $\{N_i\}_{i\in I}$ of D-dg-modules. Note that $\mathrm{P}(\mathrm{D}^{\mathrm{op}})$ is denoted by $\widehat{\mathrm{D}}_{\mathrm{pe}}$ in [TVa].

Remark 9.2.2. By Lemma 9.2.1 and the equivalence (9.2.1), we can regard $[\mathrm{Mod}_{\mathrm{dg}}(\mathrm{D})^{\circ}] \simeq \mathrm{Ho}(\mathrm{Mod}_{\mathrm{dg}}(\mathrm{D}))$ as a triangulated category. Then a perfect D-dg-module is nothing but a compact objects of this triangulated category.

The $\mathrm{C}(k)$-enriched version of the Yoneda lemma gives a quasi-fully faithful dg-functor

$$\mathrm{D} \longrightarrow \mathrm{Mod}_{\mathrm{dg}}(\mathrm{D}^{\mathrm{op}}), \quad x \longmapsto h_x.$$

For any $x \in \mathrm{D}$, the D^{op}-dg-module h_x is perfect. Thus the above dg-functor factors to

$$h : \mathrm{D} \longrightarrow \mathrm{P}(\mathrm{D}).$$

Notation 9.2.3. We call the dg-functor h the *dg Yoneda embedding.*

We close this part by recalling the notion of pseudo-perfect dg-modules [TVa, Definition 2.7]. We denote by $\mathrm{dgCat}_{\mathbb{V}}$ the category of $\mathbb{V}$-small dg-categories over k and dg-functors (Notation 9.1.3). By [Ta], it has a model structure in which weak equivalences are quasi-equivalences (Definition 9.1.5). Recall that we have a tensor product $\mathrm{D} \otimes \mathrm{D}'$ of dg-categories D and D′ with $\mathrm{Ob}(\mathrm{D} \otimes \mathrm{D}') := \mathrm{Ob}(\mathrm{D}) \times \mathrm{Ob}(\mathrm{D}')$ and

$$\mathrm{Hom}_{\mathrm{D}\otimes\mathrm{D}'}((x, y), (x', y')) := \mathrm{Hom}_{\mathrm{D}}(x, y) \otimes_k \mathrm{Hom}_{\mathrm{D}'}(y, y').$$

See also [T2, §4]. This tensor product gives rise to a derived tensor product $\otimes^{\mathbb{L}}$ in the homotopy category $\mathrm{Ho}(\mathrm{dgCat}_{\mathbb{V}})$. Here the symbol Ho denotes the homotopy category of a model category (§0.2).

Let D and D′ be two dg-categories. By applying to $\mathrm{D} \otimes^{\mathbb{L}} \mathrm{D}'$ the construction $\mathrm{P}(-)$ which is well-defined on $\mathrm{Ho}(\mathrm{dgCat}_{\mathbb{V}})$, we obtain $\mathrm{P}(\mathrm{D} \otimes^{\mathbb{L}} \mathrm{D}')$. For each object E of this $\mathrm{C}(k)$-model category, we can define a functor $F_E : \mathrm{D} \to \mathrm{Mod}_{\mathrm{dg}}(\mathrm{D}')^{\circ}$ by sending $x \in \mathrm{D}$ to

$$F_E(x) : \mathrm{D}' \longrightarrow \mathrm{C}(k), \quad y \longmapsto E(x, y).$$

Definition. An object $E \in \mathrm{P}(\mathrm{D}\otimes^{\mathbb{L}}\mathrm{D}')$ is called *pseudo-perfect relatively to* D′ if the morphism F_E factorizes through $\mathrm{P}(\mathrm{D}')$ in $\mathrm{Ho}(\mathrm{dgCat}_{\mathbb{V}})$.

If $\mathrm{D}' = k_{\mathrm{dg}}$ (Notation 9.1.2), then such E is called a *pseudo-perfect* D*-dg-module.*

9.3. **Moduli functor of perfect objects.** We continue to use the symbols in the previous §9.2. The main reference of this part is [TVa, §3].

For a commutative simplicial k-algebra $A \in \mathrm{sCom}_k$, we denote by $N(A)$ the normalized chain complex with the structure of a commutative k-dg-algebra (see §E.3 for the detail). We consider $N(A)$ as a dg-category with one object, and apply the argument in the previous §9.2 to the dg-category $\mathrm{D} = N(A)$. Then we have the dg-category

$$\mathrm{Mod}_{\mathrm{dg}}(A) := \mathrm{Mod}_{\mathrm{dg}}(N(A))$$

of dg-modules over the dg-algebra $N(A)$. We then have full sub-dg-category $\mathrm{Mod}_{\mathrm{dg}}(A)^{\circ} \subset \mathrm{Mod}_{\mathrm{dg}}(A)$ spanned by (fibrant-)cofibrant objects, and denote by

$$\mathrm{P}(A) := \mathrm{P}(N(A)) = \mathrm{Mod}_{\mathrm{dg}}(N(A))^{\circ}_{\mathrm{pe}}$$

the full sub-dg-category of perfect objects in $\mathrm{Mod}_{\mathrm{dg}}(A)^{\circ}$.

The correspondence $A \mapsto \mathrm{Mod}_{\mathrm{dg}}(A)$ induces, after a strictification procedure, a functor from sCom_k to the category of $\mathrm{C}(k)$-model categories and $\mathrm{C}(k)$-enriched left Quillen functors (see [TVa, §3.1, p.417–418] for the detail). Applying the construction $\mathrm{M} \mapsto \mathrm{M}^{\circ}$ for $\mathrm{C}(k)$-model-categories M levelwise, we obtain a functor $\mathrm{sCom}_k \to \mathrm{dgCat}_{\mathbb{V}}$, $A \mapsto \widehat{A}$. Here $\mathrm{dgCat}_{\mathbb{V}}$ denotes the model category of the $\mathbb{V}$-small dg-categories. Taking the sub-dg-category of perfect objects. we obtain a functor

$$\mathrm{sCom}_k \longrightarrow \mathrm{dgCat}_{\mathbb{V}}, \quad A \longmapsto \widehat{A}_{\mathrm{pe}}, \quad (A \to B) \longmapsto (N(B) \otimes_{N(A)} - : \widehat{A}_{\mathrm{pe}} \to \widehat{B}_{\mathrm{pe}}).$$

Next we turn to the definition of the moduli functor. Let D be a dg-category over k, and consider the following functor.

$$\mathcal{M}_{\mathrm{D}} : \mathrm{sCom}_k \longrightarrow \mathrm{Set}_{\Delta}, \quad \mathcal{M}_{\mathrm{D}}(A) := \mathit{Map}_{\mathrm{dgCat}_{\mathbb{V}}}(\mathrm{D}^{\mathrm{op}}, \mathrm{P}(A)).$$

Here Set_Δ denotes the category of simplicial sets (Definition 1.1.1) and $Map_{\mathrm{dgCat}_\mathbb{V}}$ denotes the mapping space in the model category $\mathrm{dgCat}_\mathbb{V}$, which is regarded as a simplicial set. For a morphism $A \to B$ in sCom_k, the morphism $\mathcal{M}_\mathrm{D}(A) \to \mathcal{M}_\mathrm{D}(B)$ is given by composition with $N(B) \otimes_{N(A)} - : \mathrm{P}(A) \to \mathrm{P}(B)$.

By Fact 1.1.2, the value of the functor $\mathcal{M}_\mathrm{D}$ is actually in the category of Kan complexes. Recall also that in §2.2 we set $\mathsf{dAff}_k = (\mathsf{sCom}_k)^{\mathrm{op}}$. Thus we see that $\mathcal{M}_\mathrm{D}$ determines a presheaf of spaces over dAff_k in the sense of Definition 1.5.1. We will denote the obtained presheaf by the same symbol:

$$\mathcal{M}_\mathrm{D} \in \mathsf{PSh}(\mathsf{dAff}_k) = \mathsf{Fun}((\mathsf{dAff}_k)^{\mathrm{op}}, \mathcal{S}).$$

Now let us cite

Fact ([TVa, Lemma 3.1]). The presheaf $\mathcal{M}_\mathrm{D} \in \mathsf{PSh}(\mathsf{dAff}_k)$ is a derived stack over k.

Following [TVa, Definition 3.2], we call $\mathcal{M}_\mathrm{D}$ the *moduli stack of pseudo-perfect* D^{op}*-dg-modules*.

As explained in loc. cit., the 0-th homotopy $\pi_0(\mathcal{M}_\mathrm{D}(k))$ is bijective to the set of isomorphism classes of pseudo-perfect D^{op}-dg-modules in $\mathrm{Ho}(\mathrm{Mod}_{\mathrm{dg}}(\mathrm{D}^{\mathrm{op}}))$. For each $x \in \mathrm{Ho}(\mathrm{Mod}_{\mathrm{dg}}(\mathrm{D}^{\mathrm{op}}))$, we have $\pi_1(\mathcal{M}_{\mathrm{D}^{\mathrm{op}}}, x) \simeq \mathrm{Aut}_{\mathrm{Ho}(\mathrm{Mod}_{\mathrm{dg}}(\mathrm{D}^{\mathrm{op}}))}(x,x)$ and $\pi_i(\mathcal{M}_{\mathrm{D}^{\mathrm{op}}}, x) \simeq \mathrm{Ext}^i_{\mathrm{Ho}(\mathrm{Mod}_{\mathrm{dg}}(\mathrm{D}^{\mathrm{op}}))}(x,x)$ for $i \in \mathbb{Z}_{\geq 2}$, where $\mathrm{Ho}(\mathrm{Mod}_{\mathrm{dg}}(\mathrm{D}^{\mathrm{op}}))$ is regarded as a triangulated category.

Let us cite another observation from [TVa].

Definition 9.3.1 ([TVa, Definition 2.4]). Let D be a dg-category over k.

(1) D is *proper* if the triangulated category $[\mathrm{Mod}_{\mathrm{dg}}(\mathrm{D}^{\mathrm{op}})]$ has a compact generator, and if $\mathrm{Hom}_\mathrm{D}(x,y)$ is a perfect complex of k-modules for any $x, y \in \mathrm{D}$.
(2) D is *smooth* if the $(\mathrm{D}^{\mathrm{op}} \otimes^{\mathbb{L}})$-dg-module $\mathrm{D}^{\mathrm{op}} \otimes^{\mathbb{L}} \mathrm{D} \to \mathrm{C}(k)$, $(x,y) \mapsto \mathrm{Hom}_\mathrm{D}(x,y)$ is perfect.
(3) D is *triangulated* if the dg Yoneda embedding $\mathrm{D} \to \mathrm{P}(\mathrm{D}^{\mathrm{op}})$ (Notation 9.2.3) is a quasi-equivalence.
(4) D is *saturated* if it is proper, smooth and triangulated.

Fact 9.3.2. If the dg-category D is saturated, then we have an equivalence

$$\mathcal{M}_\mathrm{D}(k) \simeq Map_{\mathrm{dgCat}}(k_{\mathrm{dg}}, \mathrm{D})$$

of simplicial sets. In particular, $\mathcal{M}_\mathrm{D}(k)$ is a model for the classifying space of objects in D.

9.4. **Geometricity of moduli stacks of perfect objects.** Now we can explain the main result in [TVa].

Definition 9.4.1 ([TVa, Definition 2.4. 7]). A dg-category D over k is *of finite type* if there exists a k-dg-algebra B which is homotopically finitely presented in the model category dgAlg_k of k-dg-algebras such that $\mathrm{P}(\mathrm{D}) = \mathrm{Mod}_{\mathrm{dg}}(\mathrm{D})^{\circ}_{\mathrm{pe}}$ is quasi-equivalent (Definition 9.1.5) to $\mathrm{Mod}_{\mathrm{dg}}(B)^{\circ}$.

Fact 9.4.2 ([TVa, Theorem 3.6]). If D is a dg-category over k of finite type, then the derived stack $\mathcal{M}_\mathrm{D}$ is locally geometric and locally of finite presentation.

Since this fact is crucial for our study, let us explain an outline of the proof. We start with the definition of Tor amplitude of dg-modules.

Definition ([TVa, Definition 2.21]). Let $A \in \mathrm{sCom}_k$, and $N(A)$ be the commutative k-dg-algebra in §9.3. A dg-module P over $N(A)$ is called *of Tor amplitude contained in* $[a,b]$ if for any (non-dg) left module M over $\pi_0(A)$ we have $H^i(P \otimes^{\mathbb{L}}_{N(A)} M) = 0$ for any $i \notin [a,b]$. Here $\otimes^{\mathbb{L}}_{N(A)}$ denotes the derived tensor product arising from the tensor product $\otimes_{N(A)}$ of dg-modules over $N(A)$.

9.4.1. Now let us explain an outline of the proof. We first consider the case $\mathrm{D} = k_{\mathrm{dg}}$, the trivial dg-category k_{dg} over k (Notation 9.1.2), which is of finite type.

The derived stack $\mathcal{M}_{k_{\mathrm{dg}}}$ can be regarded as the moduli stack of perfect k-modules. Let us define a substack $\mathcal{M}^{[a,b]}_{k_{\mathrm{dg}}}$ of $\mathcal{M}_{k_{\mathrm{dg}}}$ in the following way: For each $A \in \mathrm{sCom}_k$, define $\mathcal{M}^{[a,b]}_{k_{\mathrm{dg}}}(A)$ to be the full simplicial subset of $\mathcal{M}_{k_{\mathrm{dg}}}(A)$ spanned by connected components corresponding to perfect dg-modules over $N(A)$ of Tor amplitude contained in $[a,b]$. Then $\mathcal{M}^{[a,b]}_{k_{\mathrm{dg}}}$ is a derived stack and we have

$$\mathcal{M}_{k_{\mathrm{dg}}} = \bigcup\nolimits_{a \leq b} \mathcal{M}^{[a,b]}_{k_{\mathrm{dg}}}.$$

In case of $b = a$, we have

$$\mathcal{M}^{[a,a]}_{k_{\mathrm{dg}}} \simeq \mathcal{V}ect := \bigcup\nolimits_{r \in \mathbb{N}} \mathcal{V}ect_r,$$

where $\mathcal{V}ect_r$ is the derived stack of rank r vector bundles (Definition 2.2.32). By Fact 2.2.33, the derived stack $\mathcal{M}^{[a,a]}_{k_{\mathrm{dg}}}$ is truncated, 1-geometric, smooth and of finite presentation.

In [TVa, Proposition 3.7], it is shown that $\mathcal{M}^{[a,b]}_{k_{\mathrm{dg}}}$ is an n-geometric derived stack locally of finite presentation with $n = b - a + 1$ by induction on n. The proof is done by constructing a smooth surjection $\mathcal{U} \to \mathcal{M}^{[a,b]}_{k_{\mathrm{dg}}}$ with $\mathcal{U}$ n-geometric and locally of finite presentation, so that it gives an n-atlas of $\mathcal{M}^{[a,b]}_{k_{\mathrm{dg}}}$.

The derived stack $\mathcal{U}$ is given as follows. For a derived k-algebra $A \in \mathsf{sCom}_k$, consider the category $\mathrm{Mod}_{\mathrm{dg}}(A)^I$ of morphisms in $\mathrm{Mod}_{\mathrm{dg}}(A)$. Here $I = \Delta^1$ means the 1-simplex. This category has a model structure induced levelwise by the projective model structure on $\mathrm{Mod}_{\mathrm{dg}}(A)$. We denote by $\mathrm{Mod}_{\mathrm{dg}}(A)^{I,\mathrm{cof}} \subset \mathrm{Mod}_{\mathrm{dg}}(A)^I$ the full subcategory spanned by cofibrant objects, and define

$$\mathrm{E}(A) \subset \mathrm{Mod}_{\mathrm{dg}}(A)^{I,\mathrm{cof}}$$

to be the full subcategory spanned by those objects $Q \to R$ in $\mathrm{Mod}_{\mathrm{dg}}(A)^{I,\mathrm{cof}}$ such that $Q \in \mathcal{M}^{[a,b-1]}_{k_{\mathrm{dg}}}(A)$ and $R \in \mathcal{M}^{[b-1,b-1]}_{k_{\mathrm{dg}}}(A)$. It has a levelwise model structure. Then let $\mathrm{E}(A)_W \subset \mathrm{E}(A)$ be the subcategory of weak equivalences. Now the derived stack $\mathcal{U}$ is given by

$$\mathcal{U} : \mathsf{sCom}_A \longrightarrow \mathcal{S}, \quad A \longmapsto \mathsf{N}(\mathrm{E}(A)_W).$$

We have a natural morphism $\mathcal{U} \to \mathcal{M}^{[a,b-1]}_{k_{\mathrm{dg}}} \times \mathcal{V}ect$, and by induction hypothesis the target is $(n-1)$-geometric and locally of finite presentation. By [TVa, Sub-lemma 3.9, 3.11], we can describe the fiber of p explicitly, and find that p is (-1)-representable and locally of finite presentation. It implies that $\mathcal{U}$ is $(n-1)$-representable and locally of finite presentation.

Next we construct a morphism $\mathcal{U} \to \mathcal{M}^{[a,b]}_{k_{\mathrm{dg}}}$ of derived stacks. For each $A \in \mathsf{sCom}_k$, consider the morphism $\mathcal{U}(A) \to \mathcal{M}_{k_{\mathrm{dg}}}(A)$ sending a morphism $Q \to R$ of $N(A)$-dg-modules to its homotopy fiber. The definition of $\mathcal{U}$ yields that the homotopy fiber does belong to $\mathcal{M}^{[a,b]}_{k_{\mathrm{dg}}}(A)$. Thus we have a morphism $\mathcal{U} \to \mathcal{M}^{[a,b]}_{k_{\mathrm{dg}}}$. This is obviously $(n-1)$-representable and locally of finite presentation. A simple argument shows that it is a surjection. The proof of smoothness is non-trivial but it is done in [TVa, Lemma 3.12].

9.4.2. We turn to the case of an arbitrary dg-category D of finite type. Let us define the derived stack $\mathcal{M}^{[a,b]}_{\mathrm{D}}$ by the following cartesian square in dSt.

$$\begin{array}{ccc} \mathcal{M}^{[a,b]}_{\mathrm{D}} & \longrightarrow & \mathcal{M}^{[a,b]}_{k_{\mathrm{dg}}} \\ \downarrow & & \downarrow \\ \mathcal{M}_{\mathrm{D}} & \longrightarrow & \mathcal{M}_{k_{\mathrm{dg}}} \end{array}$$

Since $\mathcal{M}_{k_{\mathrm{dg}}} = \bigcup_{a \le b} \mathcal{M}^{[a,b]}_{k_{\mathrm{dg}}}$, we have $\mathcal{M}_{\mathrm{D}} = \bigcup_{a \le b} \mathcal{M}^{[a,b]}_{\mathrm{D}}$. In [TVa, Proposition 3.13] it is shown that the morphism $\mathcal{M}^{[a,b]}_{\mathrm{D}} \to \mathcal{M}^{[a,b]}_{k_{\mathrm{dg}}}$ is n-representable and locally of finite presentation for some n. Then one can deduce that $\mathcal{M}^{[a,b]}_{\mathrm{D}}$ is a locally geometric derived stack locally of finite presentation. Here one needs to show inductively that $\mathcal{M}^{[a,b]}_{\mathrm{D}}$ is strongly quasi-compact in the sense of [TVa, §2.3].

Let us close this subsection by the estimate on the geometricity n appearing in the last argument. Recall from the beginning of this subsection that P(D) is quasi-equivalent to $\mathrm{Mod}_{\mathrm{dg}}(B)^{\circ}$ with some k-dg-algebra B of homotopically finitely presentation. By [TVa, Proposition 2.2], B is equivalent to a retract of a dg-algebra B' which sits in a sequence

$$k = B_0 \longrightarrow B_1 \longrightarrow \cdots \longrightarrow B_m \longrightarrow B_{m+1} = B'$$

such that for each $i = 0, \ldots, m$, there exists a homotopy pushout square

$$\begin{array}{ccc} C_i & \longrightarrow & k \\ \downarrow & & \downarrow \\ B_i & \longrightarrow & B_{i+1} \end{array}$$

with C_i the free dg-algebra over the complex $k[p_i]$ for some $p_i \in \mathbb{Z}$. Then the argument in [TVa, Proposition 3.13] shows

Remark 9.4.3. We can take

$$n = b - a - \min_{0 \le i \le m} p_i.$$

9.5. **Moduli stack of complexes of quiver representations.** We cite an application of Fact 9.4.2 from [TVa, §3.5]. Let Q be a quiver and $B(Q) := kQ$ be the path algebra of Q over k. Considering $B(Q)$ as a dg-algebra over k, we have a dg-category with a unique object $B(Q)$, which will be denoted by $\mathrm{B}(Q)$. Under the notation in the previous §9.2, the dg-category $\mathrm{Mod}_{\mathrm{dg}}(\mathrm{B}(Q))^{\circ}$ can be identified with the dg-category of complexes of representations of Q over k. A pseudo-perfect object in $\mathrm{Mod}_{\mathrm{dg}}(\mathrm{B}(Q))$ is a complex of representations of Q whose underlying complex of k-modules is perfect.

If Q is finite and has no loops, then $B(Q)$ is a projective k-module of finite type, and is a perfect complex of $B(Q) \otimes_k B(Q)^{\mathrm{op}}$-modules. Moreover the dg-category $\mathrm{B}(Q)$ is smooth and proper. Now we can apply Fact 9.4.2 to the dg-category $\mathrm{B}(Q)$.

Definition. Let Q be a finite quiver with no loops. We call the derived stack $\mathcal{M}_{\mathrm{B}(Q)}$ the *derived stack of perfect complexes of representations of* Q and denote it by

$$\mathcal{P}\mathrm{f}(Q) := \mathcal{M}_{\mathrm{B}(Q)}.$$

Fact 9.5.1 ([TVa, Corollary 3.34]). Let Q be a finite quiver with no loops. Then the derived stack $\mathcal{P}\mathrm{f}(Q)$ is locally geometric and locally of finite presentation over k.

By [TVa, Example 2.5. 3], for any finite dimensional k-algebra B of finite global dimension, the dg-category B is saturated (Definition 9.3.1). In particular, $\mathrm{B}(Q)$ is saturated. Thus Fact 9.3.2 works for $\mathcal{P}\mathrm{f}(Q)$, and we can regard $\pi_0(\mathcal{P}\mathrm{f}(Q)(k))$ as the set of isomorphism classes of representations of Q over k.

Remark 9.5.2. If we consider the abelian category $\mathrm{Mod}(B(Q))$ of representations of the quiver Q instead of $\mathrm{Mod}_{\mathrm{dg}}(\mathrm{B}(Q))$, then the corresponding moduli stack is an algebraic stack locally of finite type by Remark 9.4.3,

10. Derived Hall algebra and its geometric formulation

10.1. **Ringel-Hall algebra.** In this section we give a brief account on the Ringel-Hall algebra.

We call a category A *essentially small* if the isomorphism classes of objects form a small set, which is denoted by $\mathrm{Iso}(\mathrm{A})$. For an object M of A its isomorphism class is denoted by $[M] \in \mathrm{Iso}(\mathrm{A})$.

Let $k = \mathbb{F}_q$ be a finite field with $|k| = q$. Let A be a category satisfying the following conditions.

(i) A is essentially small, abelian and k-linear.

(ii) A is of finite global dimension, and $\mathrm{Ext}^i_{\mathrm{A}}(-,-)$ has finite dimension over k for any $i \in \mathbb{N}$.

We denote by $\mathbb{Q}_c(\mathrm{A})$ the linear space of $\mathbb{Q}$-valued functions on $\mathrm{Iso}(\mathrm{A})$ with finite supports. We have a basis $\{1_{[M]} \mid [M] \in \mathrm{Iso}(\mathrm{A})\}$ of $\mathbb{Q}_c(\mathrm{A})$, where $1_{[M]}$ means the characteristic function of $[M]$. The correspondence $1_{[M]} \mapsto [M]$ gives an identification $\mathbb{Q}_c(\mathrm{A}) \xrightarrow{\sim} \bigoplus_{[M]\in \mathrm{Iso}(\mathrm{A})} \mathbb{Q}[M]$, and we will always identify these two spaces.

For $M, N, R \in \mathrm{Ob}(\mathrm{A})$, we set $\mathrm{Ext}^1_{\mathrm{A}}(M,N)_R := \{0 \to N \to R \to M \to 0 \mid \text{exact in } \mathrm{A}\}$ and

$$a_M := |\mathrm{Aut}(M)|, \quad e^R_{M,N} := \left|\mathrm{Ext}^1_{\mathrm{A}}(M,N)_R\right|, \quad g^R_{M,N} := a_M^{-1} a_N^{-1} e^R_{M,N}.$$

Note that $a_M, e^R_{M,N}, g^R_{M,N}$ are well-defined by the condition (i) above and depend only on the isomorphism classes $[M],[N],[R] \in \mathrm{Iso}(\mathrm{A})$.

For $[M],[N] \in \mathrm{Iso}(\mathrm{A})$ we define $[M] * [N] \in \mathbb{Q}_c(\mathrm{A})$ by

$$[M] * [N] := \sum_{[R]\in \mathrm{Iso}(\mathrm{A})} g^R_{M,N}[R],$$

where we choose representatives $M, N, R \in \mathrm{Ob}(\mathrm{A})$ for the fixed isomorphism classes $[M],[N],[R] \in \mathrm{Iso}(\mathrm{A})$. Denote by $[0] \in \mathrm{Iso}(\mathrm{A})$ the isomorphism class of the zero object 0 in A.

Fact 10.1.1 ([R]). The triple

$$\mathrm{Hall}(\mathrm{A}) := \big(\mathbb{Q}_c(\mathrm{A}), *, [0]\big)$$

is a unital associative $\mathbb{Q}$-algebra which has a grading with respect to the Grothendieck group $K_0(\mathrm{A})$ of A. It is called the *Ringel-Hall algebra* of A.

Let us recall another definition of $g^R_{M,N}$. For $M, N, R \in \mathrm{Ob}(\mathrm{A})$ we have

$$g^R_{M,N} = \left|\mathcal{G}^R_{M,N}\right|, \quad \mathcal{G}^R_{M,N} := \{N' \subset R \mid N' \simeq N,\ R/N \simeq M\}.$$

One can prove this statement by considering a free action of $\mathrm{Aut}(M) \times \mathrm{Aut}(N)$ on $\mathrm{Ext}^1_{\mathrm{A}}(M,N)_R$.

We have the following meaning of the multi-component product. For $M, N_1, \ldots, N_r \in \mathrm{Ob}(\mathrm{A})$, we set

$$\mathcal{F}(M; N_1, \ldots, N_r) := \{M_\bullet = (M = M_1 \supset \cdots \supset M_r \supset M_{r+1} = 0) \mid M_i/M_{i+1} \simeq N_i\ (i = 1, \ldots, r)\}.$$

Then we have
$$[N_1] * \cdots * [N_r] = \sum_{[M]\in \mathrm{Iso}(\mathrm{A})} \sum_{M_\bullet \in \mathcal{F}(M;N_1,\ldots,N_r)} [M].$$

10.2. **Derived Hall algebra.** In this subsection we recall the derived Hall algebra introduced by Toën [T1].

Let D be a small dg-category over $\mathbb{F}_q$ which is locally finite in the sense of [T1, Definition 3.1]. In other words, for any objects $x, y \in \mathrm{D}$, the complex $\mathrm{Hom}_{\mathrm{D}}(x,y)$ is cohomologically bounded and with finite-dimensional cohomology groups.

As in §9.2, we denote by $\mathrm{Mod}_{\mathrm{dg}}(\mathrm{D})$ the model category of D-dg-modules, and by
$$\mathrm{P}(\mathrm{D}) := \mathrm{Mod}_{\mathrm{dg}}(\mathrm{D})^{\circ}_{\mathrm{pe}} \subset \mathrm{Mod}_{\mathrm{dg}}(\mathrm{D})$$
the full subcategory spanned by perfect (fibrant-)cofibrant objects. The category P(D) has an induced model structure, and we can consider the homotopy category $\mathrm{Ho}\,\mathrm{P}(\mathrm{D})$. For $x, y \in \mathrm{P}(\mathrm{D})$ and $i \in \mathbb{Z}$, we denote by
$$\mathrm{Ext}^i(x,y) := \mathrm{Hom}_{\mathrm{Ho}\,\mathrm{P}(\mathrm{D})}(x, y[i])$$
where $[i]$ denotes the shift in the triangulated category $\mathrm{Ho}\,\mathrm{P}(\mathrm{D})$.

Let us consider the category $\mathrm{Mod}_{\mathrm{dg}}(\mathrm{D})^I := \mathrm{Fun}(\Delta^1, \mathrm{Mod}_{\mathrm{dg}}(\mathrm{D}))$, where $I = \Delta^1$ denotes the 1-simplex (see §1.1). This category has a model structure determined levelwise by that on $\mathrm{Mod}_{\mathrm{dg}}(\mathrm{D})$. In particular, a cofibrant object in $\mathrm{Mod}_{\mathrm{dg}}(\mathrm{D})^I$ is a morphism $u : x \to y$ in $\mathrm{Mod}_{\mathrm{dg}}(\mathrm{D})$ where x and y are cofibrant and u is a cofibration.

For an object $u : x \to y$ in $\mathrm{Mod}_{\mathrm{dg}}(\mathrm{D})^I$, we set
$$s(u) := x, \quad c(u) := y, \quad t(u) := y \coprod^{x} 0.$$
These determine left Quillen functors
$$s, c, t : \mathrm{Mod}_{\mathrm{dg}}(\mathrm{D})^I \longrightarrow \mathrm{Mod}_{\mathrm{dg}}(\mathrm{D}).$$

Next we denote by $w\,\mathrm{P}(\mathrm{D})^{\mathrm{cof}} \subset \mathrm{P}(\mathrm{D})$ the subcategory of cofibrant objects and weak equivalences between them. We define the subcategory $w(\mathrm{P}(\mathrm{D})^I)^{\mathrm{cof}} \subset \mathrm{P}(\mathrm{D})^I$ in the same way, By restricting the functors s, c, t to these subcategories, we obtain the following diagram.

(10.2.1)
$$\begin{array}{ccc} w(\mathrm{P}(\mathrm{D})^I)^{\mathrm{cof}} & \xrightarrow{\ c\ } & w\,\mathrm{P}(\mathrm{D})^{\mathrm{cof}} \\ \downarrow{\scriptstyle s\times t} & & \\ w\,\mathrm{P}(\mathrm{D})^{\mathrm{cof}} \times w\,\mathrm{P}(\mathrm{D})^{\mathrm{cof}} & & \end{array}$$

Let us then define objects $X^{(0)}(\mathrm{D})$ and $X^{(0)}(\mathrm{D})$ of $\mathcal{H}$ (Definition 1.3.2) by
$$X^{(0)}(\mathrm{D}) := \mathrm{h}\,\mathsf{N}_{\mathrm{dg}}(w\,\mathrm{P}(\mathrm{D})^{\mathrm{cof}}), \quad X^{(1)}(\mathrm{D}) := \mathrm{h}\,\mathsf{N}_{\mathrm{dg}}(w(\mathrm{P}(\mathrm{D})^I)^{\mathrm{cof}}).$$
Here $\mathsf{N}_{\mathrm{dg}}(-)$ denotes the dg-nerve functor (Definition D.3.1), and $\mathrm{h} : \mathrm{Set}_\Delta \to \mathcal{H}$ denotes the functor giving the homotopy type of a simplicial set (Definition B.2.1). Then, from the diagram (10.2.1), we obtain the following diagram in $\mathcal{H}$.
$$\begin{array}{ccc} X^{(1)}(\mathrm{D}) & \xrightarrow{\ c\ } & X^{0}(\mathrm{D}) \\ \downarrow{\scriptstyle s\times t} & & \\ X^{(0)}(\mathrm{D}) \times X^{(0)}(\mathrm{D}) & & \end{array}$$

Now let us recall

Definition. An object X of $\mathcal{H}$ is called *locally finite* if for every $x \in X$, each $\pi_i(X,x)$ is a finite group and there exists an $n \in \mathbb{N}$ such that $\pi_i(X,x) = 0$ for each $i > n$. We denote by $\mathcal{H}^{\mathrm{lf}}$ the subcategory of $\mathcal{H}$ spanned by locally finite objects.

Fact 10.2.1 ([T1, Lemma 3.2])**.** The homotopy types $X^{(0)}(\mathrm{D})$ and $X^{(1)}(\mathrm{D})$ are locally finite. Moreover, for every (homotopy) fiber F of the morphism $s \times t$, the set $\pi_0(F)$ is finite.

For $X \in \mathcal{H}^{\mathrm{lf}}$, let us denote by $\mathbb{Q}_c(X)$ the linear space of $\mathbb{Q}$-valued functions on X with finite supports.

Definition. Let $f : X \to Y$ be a morphism in $\mathcal{H}^{\mathrm{lf}}$.

(1) We define a linear map $f_! : \mathbb{Q}_c(X) \to \mathbb{Q}_c(Y)$ by

$$f_!(\alpha)(y) := \sum_{x\in\pi_0(X),\, f(x)=y} \alpha(x)\cdot \prod_{i>0}\Big(|\pi_i(X,x)|^{(-1)^i}\ |\pi_i(Y,y)|^{(-1)^{i+1}}\Big) \quad (\alpha\in\mathbb{Q}_C(X),\ y\in\pi_0(Y))$$

(2) If f has a finite fiber, then we define a linear map $f^* : \mathbb{Q}_c(Y) \to \mathbb{Q}_c(X)$ by

$$f^*(\alpha)(x) := \alpha(f(x)) \quad (\alpha\in\mathbb{Q}_C(Y),\ x\in\pi_0(X)).$$

Now we can explain the definition of the derived Hall algebra.

Fact 10.2.2 ([T1, Definition 3.3, Theorem 4.1])**.** For a locally finite dg-category D over $\mathbb{F}_q$, we set

$$\mu := c_! \circ (s\times t)^* : \mathbb{Q}_c(X^{(0)}(\mathrm{D}))\otimes\mathbb{Q}_c(X^{(0)}(\mathrm{D})) \longrightarrow \mathbb{Q}_c(X^{(0)}(\mathrm{D})).$$

Then the triple

$$\mathrm{Hall}(\mathrm{D}) := (\mathbb{Q}_c(X^{(0)}(\mathrm{D})), \mu, 0)$$

is a unital associative $\mathbb{Q}$-algebra. It is called the *derived Hall algebra* of D.

If we take D to be an abelian category A satisfying the conditions (i) and (ii) in §10.1, then the derived Hall algebra Hall(A) is nothing but the Ringel-Hall algebra in Fact 10.1.1. The grading is recovered by the Grothendieck group $K_0(\mathrm{Ho}\,\mathrm{P}(\mathrm{D}))$ of the triangulated category $\mathrm{Ho}\,\mathrm{P}(\mathrm{D}) \subset \mathrm{Ho}\,\mathrm{Mod}_{\mathrm{dg}}(\mathrm{D})$ (Remark 9.2.2).

10.3. **Geometric formulation of derived Hall algebra.** In this subsection we give the main content of this article. We set $k = \mathbb{F}_q$, the finite field of order q, and take it as the base ring for the following discussion. Let D be a dg-category over k which is of finite type. We then have the moduli stack

$$\mathcal{M}_{\mathrm{D}} : A \mapsto \mathit{Map}(\mathrm{D}^{\mathrm{op}}, \mathrm{P}(A))$$

of pseudo-perfect D-modules locally geometric and locally of finite type. By the discussion in §9.4.2, we have the stratification

$$\mathcal{M}_{\mathrm{D}} = \bigcup\nolimits_{a\le b}\mathcal{M}_{\mathrm{D}}^{[a,b]}$$

with each $\mathcal{M}_{\mathrm{D}}^{[a,b]}$ geometric and locally of finite presentation. Note also that if we assume D to be saturated, then we have a decomposition

$$\mathcal{M}_{\mathrm{D}}^{[a,b]} = \coprod\nolimits_{\alpha\in K_0(\mathrm{Ho}\,\mathrm{P}(\mathrm{D}))}\mathcal{M}_{\mathrm{D}}^{[a,b],\alpha},$$

where $K_0(\mathrm{Ho}\,\mathrm{P}(\mathrm{D}))$ denotes the Grothendieck group of the triangulated category $\mathrm{Ho}\,\mathrm{P}(D) \subset \mathrm{Ho}\,\mathrm{Mod}_{\mathrm{dg}}(D)^\circ$ (Remark 9.2.2).

Definition. We define the functor $\mathcal{E}_{\mathrm{D}} : \mathsf{sCom}_k \to \mathcal{S}$ by

$$\mathcal{E}_{\mathrm{D}}(A) := \mathit{Map}_{\mathrm{dgCat}}\big(((\mathrm{D}^{\mathrm{op}})^I)^{\mathrm{fib}}, \mathrm{P}(\mathrm{A})\big).$$

Here $(\mathrm{D}^{\mathrm{op}})^I$ denotes the $C(k)$-model category $\mathrm{Fun}(\Delta^1, \mathrm{D}^{\mathrm{op}})$, and $(-)^{\mathrm{fib}}$ denotes the full sub-dg-category of fibrant objects. We call it the *moduli stack of cofibrations in* D.

Note that we used $(-)^{\mathrm{fib}}$ on the opposite dg-category D^{op} to parametrize cofibrations in the original D. Now the argument on $\mathcal{M}_{\mathrm{D}}$ (see §9.4) works for $\mathcal{E}_{\mathrm{D}}$, and we have

Lemma 10.3.1. The presheaf $\mathcal{E}_{\mathrm{D}}$ is a derived stack over k which is locally geometric and locally of finite type.

Now we have a similar situation to §10.2. There exist morphisms

$$s, c, t : \mathcal{E}_{\mathrm{D}} \longrightarrow \mathcal{M}_{\mathrm{D}}$$

of derived stacks sending a cofibration $u : N \to M$ to $s(u) = N$, $c(u) = M$ and $t(u) = N \coprod^M 0$ respectively.

In the following we assume D is saturated. Similarly to $\mathcal{M}_{\mathrm{D}}$, the derived stack $\mathcal{E}_{\mathrm{D}}$ has a stratification

$$\mathcal{E}_{\mathrm{D}} = \bigcup\nolimits_{a\le b}\mathcal{E}_{\mathrm{D}}^{[a,b]},$$

where $\mathcal{E}_{\mathrm{D}}^{[a,b]}$ parametrizes $u : N \to M$ such that M has Tor amplitude in $[a,b]$. Each $\mathcal{E}_{\mathrm{D}}^{[a,b]}$ has a decomposition

$$\mathcal{E}_{\mathrm{D}}^{[a,b]} = \coprod\nolimits_{\alpha,\beta\in K_0(\mathrm{P}(\mathrm{D}))}\mathcal{E}_{\mathrm{D}}^{[a,b],\alpha,\beta},$$

where $\mathcal{E}_{\mathrm{D}}^{[a,b],\alpha,\beta}$ parametrizes $u : N \to M$ with $\alpha = \overline{N}$ and $\beta = \overline{t(u)}$. Here we denoted by $\overline{N}$ the class of $N \in \mathrm{P}(D)$ in $K_0(\mathrm{P}(D))$.

Then the morphisms s, t, u respect the stratification and decomposition. The restrictions of s, c, t give a diagram

$$\begin{array}{ccc} \mathcal{E}_{\mathrm{D}}^{[a,b],\alpha,\beta} & \xrightarrow{\ c\ } & \mathcal{M}_{\mathrm{D}}^{[a,b],\alpha+\beta} \\ \downarrow{\scriptstyle p} & & \\ \mathcal{M}_{\mathrm{D}}^{[a,b],\alpha} \times \mathcal{M}_{\mathrm{D}}^{[a,b],\beta} & & \end{array}$$

of derived stacks with $p := s \times t$. The following is now obvious.

Lemma 10.3.2. The morphisms s and t are smooth, and the morphism c is proper.

Now let $\overline{\mathbb{Q}}_\ell$ be the algebraic closure of the field $\mathbb{Q}_\ell$ of ℓ-adic numbers where ℓ and q are assumed to be coprime. We can apply the construction in §7 to the present situation, and have the derived ∞-category $\mathbf{D}_{\infty,\mathrm{c}}(\mathcal{M}_{\mathrm{D}}, \overline{\mathbb{Q}_\ell})$ of ℓ-adic constructible sheaves. We denote it by

$$\mathbf{D}_{\infty,\mathrm{c}}(\mathcal{M}_{\mathrm{D}}) := \mathbf{D}_{\infty,\mathrm{c}}(\mathcal{M}_{\mathrm{D}}, \overline{\mathbb{Q}_\ell}).$$

The stratification and decomposition of $\mathcal{M}_{\mathrm{D}}$ gives $\mathbf{D}_{\infty,\mathrm{c}}(\mathcal{M}_{\mathrm{D}}) = \sum_{a \le b} \oplus_\alpha \mathbf{D}_{\infty,\mathrm{c}}(\mathcal{M}_{\mathrm{D}}^{[a,b],\alpha})$. By Lemma 10.3.2, we have the derived functors

$$p^* : \mathbf{D}_{\infty,\mathrm{c}}(\mathcal{M}_{\mathrm{D}}^\alpha \times \mathcal{M}_{\mathrm{D}}^\beta) \longrightarrow \mathbf{D}_{\infty,\mathrm{c}}(\mathcal{E}_{\mathrm{D}}^{\alpha,\beta}), \quad c_! : \mathbf{D}_{\infty,\mathrm{c}}(\mathcal{E}_{\mathrm{D}}^{\alpha,\beta}) \longrightarrow \mathbf{D}_{\infty,\mathrm{c}}(\mathcal{M}_{\mathrm{D}}^{\alpha+\beta}).$$

Here we suppressed Tor amplitude $[a, b]$ in the superscripts. We also have

$$\mathbf{D}_{\infty,\mathrm{c}}(\mathcal{M}_{\mathrm{D}}^\alpha \times \mathcal{M}_{\mathrm{D}}^\beta) \simeq \mathbf{D}_{\infty,\mathrm{c}}(\mathcal{M}_{\mathrm{D}}^\alpha, \overline{\mathbb{Q}_\ell}) \times \mathbf{D}_{\infty,\mathrm{c}}(\mathcal{M}_{\mathrm{D}}^\beta),$$

where the product in the right hand side denotes the product of simplicial sets. Now we can introduce

Definition 10.3.3. For $\alpha, \beta \in K_0(\mathrm{P}(\mathrm{D}))$, we define a functor $\mu_{\alpha,\beta}$ by

$$\mu_{\alpha,\beta} : \mathbf{D}_{\infty,\mathrm{c}}(\mathcal{M}_{\mathrm{D}}^\alpha) \times \mathbf{D}_{\infty,\mathrm{c}}(\mathcal{M}_{\mathrm{D}}^\beta) \longrightarrow \mathbf{D}_{\infty,\mathrm{c}}(\mathcal{M}_{\mathrm{D}}^{\alpha+\beta}), \quad M \longmapsto c_! p^*(M)[\dim p]$$

They determine a functor

$$\mu : \mathbf{D}_{\infty,\mathrm{c}}(\mathcal{M}_{\mathrm{D}}) \times \mathbf{D}_{\infty,\mathrm{c}}(\mathcal{M}_{\mathrm{D}}) \longrightarrow \mathbf{D}_{\infty,\mathrm{c}}(\mathcal{M}_{\mathrm{D}}).$$

Proposition 10.3.4. μ is associative. In other words, we have in each component an equivalence

$$\mu_{\alpha,\beta+\gamma} \circ (\mathrm{id} \times \mu_{\beta,\gamma}) \simeq \mu_{\alpha+\beta,\gamma} \circ (\mu_{\alpha,\beta} \times \mathrm{id})$$

which is unique up to contractible ambiguity. Here we suppressed Tor amplitude again.

Proof. The following argument is standard, but let us write it down for completeness. We follow the "rough part" of the proof of [S2, Proposition 1.9] both for the argument and the symbols.

The functor $\mu_{\alpha,\beta+\gamma} \circ (\mathrm{id} \times \mu_{\beta,\gamma})$ in the left hand side corresponds to the rigid line part of the following diagram.

$$\begin{array}{ccccc} \mathcal{E}^{\alpha,(\beta,\gamma)} & \overset{p_2''}{\dashrightarrow} & \mathcal{E}^{\alpha,\beta+\gamma} & \xrightarrow{\ p_2'\ } & \mathcal{M}^{\alpha+\beta+\gamma} \\ {\scriptstyle p_1''}\vdots & & \downarrow{\scriptstyle p_1'} & & \\ \mathcal{M}^\alpha \times \mathcal{E}^{\beta,\gamma} & \xrightarrow[p_2]{} & \mathcal{M}^\alpha \times \mathcal{M}^{\beta+\gamma} & & \\ \downarrow{\scriptstyle p_1} & & & & \\ \mathcal{M}^\alpha \times \mathcal{M}^\beta \times \mathcal{M}^\gamma & & & & \end{array}$$

Here we suppressed the symbol D in the subscripts. We can complete it into a commutative diagram by defining

$$\mathcal{E}^{\alpha,(\beta,\gamma)} := (\mathcal{M}^\alpha \times \mathcal{E}^{\beta,\gamma}) \times_{\mathcal{M}^\alpha \times \mathcal{M}^{\beta+\gamma}} \mathcal{E}^{\alpha,\beta+\gamma},$$

which is the moduli stack of pairs of cofibrations $(N \to M, M \to L)$ with $\overline{N} = \gamma$, $\overline{M} = \beta+\gamma$ and $\overline{L} = \alpha+\beta+\gamma$. Using the smooth base change (Proposition 7.5.1) in the completed diagram, we have

$$\begin{aligned} \mu_{\alpha,\beta+\gamma} \circ (\mathrm{id} \times \mu_{\beta,\gamma}) &\simeq (p_2')_!(p_1')^*(p_2)_!(p_1)^*[\dim p_1 + \dim p_1'] \\ &\simeq (p_2')_!(p_2'')_!(p_1'')^*(p_1)^*[\dim p_1 + \dim p_1''] \simeq (p_2' p_2'')_!(p_1 p_1'')^*[\dim(p_1 p_1'')]. \end{aligned} \tag{10.3.1}$$

In the same way, the functor $\mu_{\alpha,\beta+\gamma} \circ (\mathrm{id} \times \mu_{\beta,\gamma})$ in the right hand side corresponds to the rigid line part of

$$\begin{array}{ccccc}
\mathcal{E}^{(\alpha,\beta),\gamma} & \overset{p_2''}{\dashrightarrow} & \mathcal{E}^{\alpha+\beta,\gamma} & \overset{p_2'}{\longrightarrow} & \mathcal{M}^{\alpha+\beta+\gamma} \\
{\scriptstyle p_1''}\,\vdots & & \downarrow{\scriptstyle p_1'} & & \\
\mathcal{E}^{\alpha,\beta} \times \mathcal{M}^{\gamma} & \underset{p_2}{\longrightarrow} & \mathcal{M}^{\alpha+\beta} \times \mathcal{M}^{\gamma} & & \\
{\scriptstyle p_1}\downarrow & & & & \\
\mathcal{M}^{\alpha} \times \mathcal{M}^{\beta} \times \mathcal{M}^{\gamma} & & & &
\end{array}$$

The completion is done with

$$\mathcal{E}^{(\alpha,\beta,)\gamma} := (\mathcal{E}^{\alpha,\beta} \times \mathcal{M}^{\gamma}) \times_{\mathcal{M}^{\alpha+\beta} \times \mathcal{M}^{\gamma}} \mathcal{E}^{\alpha+\beta,\gamma},$$

which is the moduli stack of pairs of cofibrations $(R \to L \coprod^{M} 0, M \to L)$ with $\overline{M} = \gamma$, $\overline{R} = \beta$ and $\overline{L} = \alpha + \beta + \gamma$. In this case the smooth base change yields the same calculation as (10.3.1).

Thus the conclusion holds if the derived stacks $\mathcal{E}^{\alpha,(\beta,\gamma)}$ and $\mathcal{E}^{(\alpha,\beta,)\gamma}$ are equivalent. But this is shown in [T1] on the k-rational point level. The equivalence as derived stacks follows from the moduli property. □

Summarizing the arguments so far, we have

Theorem 10.3.5. Let D be a dg-category over $k = \mathbb{F}_q$ which is of finite type and saturated. Assume that the positive integer ℓ is prime to q. Then the derived ∞-category $\mathbf{D}_{\infty,c}(\mathcal{M}_{\mathrm{D}}, \overline{\mathbb{Q}_\ell})$ of ℓ-adic constructible sheaves on the moduli stack $\mathcal{M}_{\mathrm{D}}$ of pseudo-perfect D^{op}-modules has a unital associative ring structure with respect to the bifunctor μ (Definition 10.3.3).

Hereafter we denote

$$\mathcal{M} \star \mathcal{N} := \mu(\mathcal{M}, \mathcal{N}).$$

We use the ordinary symbol for an iterated multiplication: $\mathcal{M}_1 \star \cdots \star \mathcal{M}_n := \mathcal{M}_1 \star (\mathcal{M}_2 \star (\cdots \star \mathcal{M}_n))$.

10.4. **Lusztig sheaves in the derived setting.** In this subsection we focus on the case where D is given by the path algebra of a quiver. More precisely speaking, we consider the situation in §9.5. Here is a list of the notations.

- Q denotes a finite quiver without loops.
- $B(Q) := (kQ)^{\mathrm{op}}$ denotes the opposite of the path algebra of Q over $k = \mathbb{F}_q$.
- $\mathrm{B}(Q)$ denotes the dg-category with a unique object $B(Q)$.

Denoting by $\mathrm{Rep}(Q)$ the category of representation of Q, we know that $\mathrm{Rep}(Q)$ is a hereditary abelian category, i.e., its global dimension is ≤ 1. We also know that $K_0(\mathrm{P}(\mathrm{B}(Q))) = K_0(\mathrm{Rep}(Q)) = \mathbb{Z}^{Q_0}$ with Q_0 the set of vertices in Q. Its element is called a dimension vector.

Let us denote by

$$\mathcal{P}\mathrm{f}(Q) := \mathcal{M}_{\mathrm{B}(Q)}$$

the moduli stack of perfect complexes of representations of Q. It has a stratification and decomposition

$$\mathcal{P}\mathrm{f}(Q) = \bigcup\nolimits_{a \leq b} \mathcal{P}\mathrm{f}(Q)^{[a,b]}, \quad \mathcal{P}\mathrm{f}(Q)^{[a,b]} = \coprod\nolimits_{\alpha \in \mathbb{Z}^{Q_0}} \mathcal{P}\mathrm{f}(Q)^{[a,b],\alpha}.$$

Hereafter we denote

$$\mathcal{P}\mathrm{f}(Q)^{n,\alpha} := \mathcal{P}\mathrm{f}(Q)^{[n,n],\alpha}.$$

Recalling that $\mathrm{B}(Q)$ is of finite type and saturated, we can apply Theorem 10.3.5 to $\mathrm{D} = \mathrm{B}(Q)$. Then we have an associative ring structure on $\mathbf{D}_{\infty,c}(\mathcal{P}\mathrm{f}(Q), \overline{\mathbb{Q}_\ell})$, which is a geometric version of the derived Hall algebra for Q.

The ∞-category $\mathbf{D}_{\infty,c}(\mathcal{P}\mathrm{f}(Q), \overline{\mathbb{Q}_\ell})$ is quite large, and we should consider only an "accessible" part of it. Following the non-derived case established by Lusztig (see [Lus, Chap. 9] for example), we consider the sub-∞-category generated by constant perverse sheaves. We also follow [S2, §1.4] for the argument and the symbols.

For each $\alpha \in \mathbb{N}^{Q_0}$ and $s \in \mathbb{Z}$, let

$$\mathbb{1}_{\alpha,s} := (\overline{\mathbb{Q}_\ell})_{\mathcal{P}\mathrm{f}(Q)^{n,\alpha}}[\dim \mathcal{P}\mathrm{f}(Q)^{s,\alpha} + s] \in \mathbf{D}^{b}_{\infty,c}(\mathcal{P}\mathrm{f}(Q), \overline{\mathbb{Q}_\ell})$$

be the shifted constant $\overline{\mathbb{Q}_\ell}$-sheaf on $\mathcal{P}\mathrm{f}(Q)^{s,\alpha}$. For a sequence $(\alpha_1,\ldots,\alpha_m)$ with $\alpha_j \in \mathbb{N}^{Q_0}$ and another $(s_1,\ldots,s_m)$ with $s_j \in \mathbb{Z}$, we define

$$L^{(s_1,\ldots,s_m)}_{(\alpha_1,\ldots,\alpha_m)} := \mathbb{1}_{\alpha_1,s_1} \star \cdots \star \mathbb{1}_{\alpha_m,s_m} \in \mathbf{D}^b_{\infty,\mathrm{c}}(\mathcal{P}\mathrm{f}(Q),\overline{\mathbb{Q}_\ell}).$$

We call it a *Lusztig sheaf.*

We call $\alpha \in \mathbb{Z}^{Q_0}$ *simple* if $\alpha = \varepsilon_i$ with some $i \in Q_0$, i.e., α has value 1 at $i \in Q_0$ and 0 at other $j \in Q_0$. By Proposition 8.3.1, the Lusztig sheaf L^s_α is a simple perverse sheaf. For a fixed $\gamma \in \mathbb{Z}^{Q_0}$, we denote by $\mathcal{P}^\gamma$ the collection of all simple perverse sheaves on $\bigcup_{a\le b} \mathcal{P}\mathrm{f}(Q)^{[a,b],\gamma}$ arising, up to shift, as a direct summand of $L^{(s_1,\ldots,s_m)}_{(\alpha_1,\ldots,\alpha_m)}$ with $\alpha_1+\cdots+\alpha_n = \gamma$ and α_i simple for all i. We also denote by $\mathsf{Q}^\gamma \subset \mathbf{D}^b_{\infty,\mathrm{c}}(\mathcal{P}\mathrm{f}(Q),\overline{\mathbb{Q}_\ell})$ the full sub-∞-category spanned by those objects which are equivalent to a direct sum $P_1 \oplus \cdots \oplus P_l$ with $P_j \in \mathcal{P}^\gamma$, and set $\mathsf{Q} := \sqcup_{\gamma\in\mathbb{Z}^{Q_0}} \mathsf{Q}^\gamma$. We call it the *derived Hall category.*

Lemma. Q is preserved by the bifunctor μ.

Proof. By the definition of Lusztig sheaves we have

$$L^{(s_1,\ldots,s_l)}_{(\alpha_1,\ldots,\alpha_l)} \star L^{(t_1,\ldots,t_m)}_{(\beta_1,\ldots,\beta_m)} \simeq L^{(s_1,\ldots,s_l),(t_1,\ldots,t_m)}_{(\alpha_1,\ldots,\alpha_l,\beta_1,\ldots,\beta_m)}.$$

□

Remark. If we restrict s_i's to be zero, then by Remark 9.5.2 the related moduli stacks are algebraic stacks, so that we recover the *Hall category* in the sense of [S2, §1.4]. It has more properties such as the existence of a coproduct and a Hopf pairing. We refer to loc. cit. and [Lus] for the detail.

Appendix A. Algebraic stacks

In this appendix we recollect some basics on algebraic stacks in the sense of [LM, O2].

A.1. **Algebraic spaces.** We begin with the recollection of algebraic spaces. Our main reference is [O2, §5.1] and [K]. We will use the notion of Grothendieck topologies and sites freely. For a scheme S, we denote by Sch_S the category of S-schemes. Let us fix the notation of the big étale topology since it will be used repeatedly.

Definition A.1.1. Let S be a scheme. The *big étale site* $\mathrm{ET}(S)$ over S is defined to be the site $\mathrm{ET}(S) := (\mathrm{Sch}_S, \mathrm{ET})$ consisting of

- The category Sch_S of S-schemes.
- The big étale topology ET.

The Grothendieck topology ET is determined as follows: the set $\mathrm{Cov}_{\mathrm{ET}}(U)$ of coverings of $U \in \mathrm{Sch}_S$ consists of a family $\{U_i \to U\}_{i\in I}$ of morphisms in Sch_S for which each $U_i \to U$ is étale and $\coprod_{i\in I} U_i \to U$ is surjective.

Recall that given a site $\mathrm{S} = (\mathrm{C}, \tau)$ we have the notion of a *sheaf* (*of sets*) *on* S (see [O2, Chap. 2] for example). It is a functor $\mathrm{C}^{\mathrm{op}} \to \mathrm{Set}$ satisfying the sheaf condition with respect to the Grothendieck topology τ. Recall also that a morphism of sheaves on S is defined to be a morphism of functors.

Given a scheme S and an S-scheme $T \in \mathrm{Sch}_S$, the functor $h_T(-) := \mathrm{Hom}_{\mathrm{Sch}_S}(-,T)$ defines a sheaf of sets on the big étale site $\mathrm{ET}(S)$. Hereafter we identify $T \in \mathrm{Sch}_S$ and the sheaf h_T on $\mathrm{ET}(S)$.

Definition. Let S be a scheme. An *algebraic space* over S is a functor $X : (\mathrm{Sch}_S)^{\mathrm{op}} \to \mathrm{Set}$ satisfying the following three conditions.

- X is a sheaf on the big étale site $\mathrm{ET}(S)$.
- The diagonal map $X \to Y := X \times_S X$ is represented by schemes, i.e., the fiber product $X \times_Y T$ is a scheme for any $T \in \mathrm{Sch}_S$ and any morphism $T \to Y$ of sheaves on $\mathrm{ET}(S)$.
- There exists $U \in \mathrm{Sch}_S$ and a morphism $U \to X$ of sheaves on $\mathrm{ET}(S)$ such that the morphism $p_T : U \times_X T \to T$ of schemes is surjective and étale for any $T \in \mathrm{Sch}_S$ and any morphism $T \to X$ of sheaves.

In this case we call either the scheme U or the morphism $U \to X$ an *étale covering* of X.

A *morphism* of algebraic spaces over S is a morphism of functors. The category of algebraic spaces over S is denoted by AS_S.

Note that a scheme $U \in \mathrm{Sch}_S$ can be naturally regarded as an algebraic space over S.

Next we recall the description of an algebraic space as a quotient.

Fact. Let X be an algebraic space over a scheme S and $U \to X$ be an étale covering.

(1) The fiber product $R := U \times_X U$ in the category of sheaves is representable by an S-scheme. The corresponding scheme is denoted by the same symbol R.
(2) The natural projections $p_1, p_2 : R \to U$ makes R into an étale equivalence relation [O2, Definition 5.2.1].
(3) The natural morphism $U/R \to X$ from the quotient sheaf U/R is an isomorphism.

We denote this situation as $R \rightrightarrows U \to X$.

Using this quotient description we have a concrete way to give a morphism of algebraic spaces.

Fact. Let X_1 and X_2 be algebraic spaces over a scheme S. Take étale coverings $U_i \to X_i$ and denote $R_i := U_i \times_{X_i} U_i$ $(i = 1, 2)$.

(1) Assume that we have a diagram

$$\begin{array}{ccccc} R_1 & \underset{p_{1,2}}{\overset{p_{1,1}}{\rightrightarrows}} & U_1 & \longrightarrow & X_1 \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\ R_2 & \underset{p_{2,2}}{\overset{p_{2,1}}{\rightrightarrows}} & U_2 & \longrightarrow & X_2 \end{array}$$

of sheaves on $\mathrm{ET}(S)$ with $h \circ p_{1,i} = p_{2,i} \circ g$ $(i = 1, 2)$. Then there is a unique morphism $f : X_1 \to X_2$ making the resulting diagram commutative.
(2) Conversely, every morphism $X_1 \to X_2$ of algebraic spaces arises in this way for some choice of U_1, U_2, g, h.

Using properties of schemes and their morphisms, we can define properties of algebraic spaces and their morphisms. We refer [O2, Definition 5.1.3] for the definition of *stability* and *locality on domain* of a property of morphisms in a site S.

Definition A.1.2. Let S be a scheme, $f : X \to Y$ be a morphism of algebraic spaces over S, and $\mathbf{P}$ be a property of morphisms in the big étale site $\mathrm{ET}(S)$.

(1) Assume $\mathbf{P}$ is stable. Then we say X *has property* $\mathbf{P}$ if it has an étale covering U whose structure morphism $U \to S$ has property $\mathbf{P}$.
(2) Assume $\mathbf{P}$ is stable and local on domain. We say f *has* $\mathbf{P}$ if there exist étale coverings $u : U \to X$ and $v : V \to Y$ such that the morphism $p_V : U \times_{f \circ u, Y, v} V \to V$ in Sch_S has property $\mathbf{P}$.

By [O2, Proposition 5.1.4] and [K, Chap. 2], the following properties $\mathbf{P}_1$ of morphisms are stable for the big étale site $\mathrm{ET}(S)$.

$$\mathbf{P}_1 := \text{separated, universally closed, quasi-compact.}$$

By loc. cit., the following properties $\mathbf{P}_2$ are stable and local on domain.

$$\mathbf{P}_2 := \text{surjective, étale, locally of finite type, smooth, universally open,}$$
$$\text{locally of finite presentation, locally quasi-finite.}$$

Note that for an étale covering $f : U \to X$ of an algebraic scheme X the morphism f is étale and surjective in the sense of Definition A.1.2.

In the main text we need the *properness* of a morphism of algebraic spaces. In order to define it, we need to introduce some classes of morphisms between algebraic spaces which are not given in Definition A.1.2. Our main reference is [K, Chap. 2]. Let us omit to mention the base scheme S in the remaining part.

Definition A.1.3 ([K, Chap. 2, Definition 1.6])**.** A morphism $f : X \to Y$ of algebraic spaces is *quasi-compact* if for any étale morphism $U \to Y$ (Definition A.1.2 (2)) with U a quasi-compact scheme the fiber product $X \times_Y U$ is a quasi-compact algebraic space (Definition A.1.2 (1)).

Definition A.1.4 ([K, Definition 3.3])**.** A morphism f of algebraic spaces is *of finite type* (resp. *of finite presentation*) if it is locally of finite type (resp. locally of finite presentation) in the sense of Definition A.1.2 (2) and quasi-compact (Definition A.1.3).

Definition A.1.5 ([K, Chap. 2, Extension 3.8, Definition 3.9])**.** Let $f : X \to Y$ be a morphism of algebraic spaces over a scheme S.

(1) f is a *closed immersion* if for any $U \in \mathrm{Sch}_S$ and any $U \to Y$ the fiber product $X \times_Y U$ is a scheme and $X \times_Y U \to U$ is a closed immersion in Sch_S. In this case, X is called a *closed subspace* of Y.
(2) f is *separated* if the induced morphism $X \to X \times_Y X$ is a closed immersion in the sense of (1).

Although it is not directly related to proper morphisms, let us introduce now the notion of *quasi-separated morphisms*.

Definition A.1.6 ([O2, Propoistion 5.4.7])**.** A morphism $f : X \to Y$ of algebraic spaces over a scheme S is *quasi-separated* if the diagonal morphism $\Delta_{X/Y} : X \to X \times_{f,Y,f} X$ is quasi-compact (Definition A.1.3). An algebraic space X over S is *quasi-separated* if the structure morphism $X \to S$ is quasi-separated.

Going back to proper morphisms, we introduce

Definition A.1.7 ([K, Chap. 2, Definition 6.1, 6.9])**.** Let X be an algebraic space.

(1) A *point* of X is a morphism $i : \operatorname{Spec} k \to X$ of algebraic spaces where k is a field and i is a categorical monomorphism. Two points $i_j : p_j \to X$ $(j = 1, 2)$ are *equivalent* if there exists an isomorphism $e : p_1 \to p_2$ such that $i_2 \circ e = i_1$.

(2) The *underlying topological space* $|X|$ of X is defined to be the set of points of X modulo equivalence together with the topology where a subset $C \subset |X|$ is closed if C is of the form $|Y|$ for some closed subspace $Y \subset X$ (Definition A.1.5 (1))

Note that a morphism $X \to Y$ of algebraic spaces naturally induces a continuous map $|X| \to |Y|$ of the underlying topological spaces.

Definition A.1.8 ([K, Chap. 2, Definition 6.9])**.** A morphism $f : X \to Y$ of algebraic spaces is *universally closed* if for any morphism $Z \to Y$ of algebraic spaces the induced continuous map $|X \times_Y Z| \to |Z|$ is closed.

Let us also recall the notion of relaitive dimension.

Definition A.1.9. Let $f : X \to Y$ be a morphism of algebraic spaces over a scheme S, and let $d \in \mathbb{N} \cup \{\infty\}$.

(1) For $x \in |X|$, f has *relative dimension* d *at the point* x if for any commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & V \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ X & \xrightarrow[f]{} & Y \end{array} \qquad \begin{array}{ccc} u & \longmapsto & v \\ \downarrow & & \downarrow \\ x & \longmapsto & y \end{array}$$

in Sch_S with a and b étale, the dimension of the local ring $\mathcal{O}_{U_v,u}$ of the fiber U_v at u is d.

(2) f has *relative dimension* d if it has relative dimension d at any $x \in |X|$.

Finally we have

Definition A.1.10 ([K, Chap. 2, Definition 7.1])**.** A morphism $f : X \to Y$ of algebraic spaces is *proper* if it is separated (Definition A.1.5 (2)), of finite type (Definition A.1.4) and universally closed (Definition A.1.8).

A.2. **Algebraic stacks.** Next we recall the definition of algebraic stacks. We assume some basics on categories fibered in groupoids [O2, Chap. 3].

Let S be a site. In order to make terminology clear, let us call a stack in the ordinary sense ([LM, Chap. 2], [O2, Chap. 4]) an *ordinary stack*. In other words, we have

Definition. Let S be a site. An *ordinary stack* over S is a category F fibered in groupoids over S such that for any object $U \in \mathrm{S}$ and any covering $\{U_i \to U\}_{i \in I}$ of U the functor $\mathrm{F}(U) \to \mathrm{F}(\{U_i \to U\}_{i \in I})$ is an equivalence of categories.

Ordinary stacks over S form a 2-category.

Definition A.2.1. An *ordinary stack* X over a scheme S means an ordinary stack X over $\mathrm{ET}(S)$, where $\mathrm{ET}(S)$ is the big étale site on S (Definition A.1.1).

A scheme and an algebraic space over S can be naturally considered as an ordinary stack over S. We further introduce

Definition A.2.2. Let S be a scheme and $f : X \to Y$ be a 1-morphism of ordinary stacks over S.

(1) f is called *representable* if the fiber product $X \times_{f,Y,g} U$ is an algebraic space over S for any $U \in \mathrm{Sch}_S$ and any 1-morphism $g : U \to Y$.

(2) f is called *quasi-compact* (resp. *separated*) if it is representable and for any $U \in \mathrm{Sch}_S$ and any 1-morphism $g : U \to Y$ the algebraic space $X \times_{f,Y,g} U$ is quasi-compact (resp. separated) in the sense of Definition A.1.2.

Here is the definition of an algebraic stack.

Definition A.2.3. An *algebraic stack over a scheme* S is defined to be an ordinary stack X over S satisfying the following two conditions.

- The diagonal 1-morphism $\Delta : X \to X \times_S X$ is representable, quasi-compact and separated in the sense of Definition A.2.2.
- There exists an algebraic space U over S and a smooth surjection $U \to X$, i.e., for any $T \in \mathrm{Sch}_S$ and any 1-morphism $T \to X$ the morphism $p_T : U \times_X T \to T$ of algebraic spaces (see Remark A.2.4 (1) below) is a smooth surjection in the sense of Definition A.1.2.

In this case the algebraic space U is called a *smooth covering* of X.

Remark A.2.4. (1) The fiber product $U \times_X T$ in the second condition is an algebraic space. In order to see this, note first that any 1-morphism $t : T \to X$ from a scheme is representable. Indeed, for any 1-morphism $u : U \to X$ from a scheme, the fiber product $U \times_{u,X,t} T$ is isomorphic to the fiber product Y of the diagram $X \xrightarrow{\Delta} X \times_S X \xleftarrow{u \times_S t} U \times_S T$. This fiber product Y is an algebraic space since Δ is representable by the first condition.

(2) One can replace the algebraic space U by a scheme which is étale covering of U. The definition in [O2, Definition 8.1.4] is given under this replacement.

Let us recall a criterion of an algebraic stack being an algebraic space.

Fact A.2.5 ([LM, Chap. 2, Proposition (4.4)])**.** An algebraic stack X over a scheme S is an algebraic space if and only if the following two conditions are satisfied.

(i) X is a Deligne-Mumford stack, i.e., there is an étale surjective 1-morphism $U \to X$ from a scheme U.
(ii) The diagonal 1-morphism $X \to X \times_S X$ is a monomorphism.

We can introduce properties of algebraic stacks as the case of algebraic spaces. We use stability and locality on domain [O2, Definition 5.1.3] of a property of morphisms in the smooth site.

Definition ([O2, Definition 2.1.16])**.** Let S be a scheme. The *smooth site* $\mathrm{Sm}(S)$ over S is defined to be the site $\mathrm{Sm}(S) := (\mathrm{Sch}_S^{\mathrm{sm}}, \mathrm{sm})$ consisting of the following data.

- The full subcategory $\mathrm{Sch}_S^{\mathrm{sm}} \subset \mathrm{Sch}_S$ spanned by smooth schemes U over S.
- The smooth topology sm.

The Grothendieck topology sm is determined as follows: the set $\mathrm{Cov}_{\mathrm{sm}}(U)$ of coverings of $U \in \mathrm{Sch}_S^{\mathrm{sm}}$ consists of a family $\{U_i \to U\}_{i \in I}$ of morphisms in $\mathrm{Sch}_S^{\mathrm{sm}}$ for which each $U_i \to U$ is smooth and the morphism $\coprod_{i \in I} U_i \to U$ is surjective.

Definition A.2.6 ([O2, Definition 8.2.1])**.** Let S be a scheme, and $\mathbf{Q}_0$ be a property of S-schemes which is stable in $\mathrm{Sm}(S)$. An algebraic stack X over S *has property* $\mathbf{Q}_0$ if there exists a smooth covering $U \to X$ from a scheme U having property $\mathbf{Q}_0$.

Let us remark that the scheme U in this definition can be replaced by an algebraic space U [O2, Lemma 8.2.4]. We can apply this definition to

$$\mathbf{Q}_0 := \text{locally noetherian, locally of finite type over } S, \text{ locally of finite presentation over } S.$$

Let us also introduce some classes of morphisms between algebraic stacks.

Definition A.2.7 ([O2, Definition 8.2.6])**.** Let S be a scheme, and $\mathbf{Q}_1$ be a property of morphisms of schemes which is stable and local on domain with respect to $\mathrm{Sm}(S)$. A 1-morphism $f : X \to Y$ of algebraic stacks *has property* $\mathbf{Q}_1$ if there exists a commutative diagram

$$\begin{array}{ccccc} U & \xrightarrow{g} & X \times_Y V & \xrightarrow{f'} & V \\ & \searrow^{q} & \downarrow & & \downarrow p \\ & & X & \xrightarrow{f} & Y \end{array}$$

of 1-morphisms between ordinary stacks where U and V are schemes, the square is cartesian, and g and p give smooth coverings such that the morphism $f' \circ g : U \to V$ of schemes has property $\mathbf{Q}_1$.

This definition is independent of the choice of smooth coverings $U \to X$ and $V \to Y$ [O2, Proposition 8.2.8]. We can apply this definition to

$$\mathbf{Q}_1 = \text{smooth, locally of finite type, locally of finite presentation, surjective.}$$

Definition A.2.8 ([O2, Definition 8.2.9])**.** Let S be a scheme, and $\mathbf{Q}_2$ be a property of morphisms of algebraic spaces over S which is stable with respect to the smooth topology on AS_S. A representable 1-morphism $f : X \to Y$ of algebraic stacks over S *has property* $\mathbf{Q}_2$ if for every 1-morphism $V \to Y$ from some $Y \in \mathrm{AS}_S$, the morphism $X \times_Y V \to V$ in AS_S has property $\mathbf{Q}_2$.

In particular, we can apply this definition to

$$\mathbf{Q}_2 = \text{étale, smooth of relative dimension } d, \text{ quasi-compact,} \\ \text{quasi-separate, proper, being a closed immersion.}$$

Since the diagonal 1-morphism $\Delta_{X/Y} : X \to X \times_{f,Y,f} Y$ is representable for any 1-morphism $f : X \to Y$ of algebraic stacks over S, we can apply Definition A.2.2 to introduce

Definition A.2.9 ([O2, Definition 8.2.12])**.** Let $f : X \to Y$ be a 1-morphism of algebraic stacks.

(1) f is *separated* if $\Delta_{X/Y}$ is proper in the sense of Definition A.2.8.
(2) f is *quasi-separated* if $\Delta_{X/Y}$ is quasi-compact and quasi-separated in the sense of Definition A.2.8.

We finally introduce

Definition A.2.10. A 1-morphism f of algebraic stacks is *of finite type* (resp. *of finite presentation*) if it is quasi-compact (Definition A.2.8) and locally of finite type (resp. locally of finite presentation) in the sense of Definition A.2.9.

A.3. **Lisse-étale site on algebraic stacks.** We cite from [LM, Chap. 12] the definition of the lisse-étale site on algebraic stacks.

Definition A.3.1. The *lisse-étale site* on an algebraic stack X over a scheme S is given by

- An object of the underlying category is a pair (U, u) of an algebraic space U over S and a 1-morphisms $u : U \to X$ of ordinary stacks over S. A morphism from $(U, u : U \to X)$ to $(V, v : V \to X)$ is a pair (φ, α) of a smooth 1-morphism $\varphi : U \to V$ of algebraic spaces over S and a 2-isomorphism $\alpha : u \xrightarrow{\sim} v \circ \varphi$.
- As for the Grothendieck topology, the set $\mathrm{Cov}(u)$ of covering sieves consists of families $\{(\varphi_i, \alpha_i) : (U_i, u_i) \to (U, i)\}_{i \in I}$ such that the 1-morphism $\coprod_{i \in I} \varphi_i : \coprod_{i \in I} U_i \to U$ of algebraic spaces over S is étale and surjective.

The associated topos is denoted by $X_{\text{lis-et}}$.

As shown in [LM, Lemme (12.1.2)], one can replace "an algebraic space U over S" in the above definition by "an affine scheme U over S", and replace "families $\{(\varphi_i, \alpha_i)\}_{i \in I}$" by "finite families $\{(\varphi_i, \alpha_i)\}_{i \in I}$". The resulting topos is equivalent to $X_{\text{lis-et}}$.

Appendix B. ∞-categories

In this appendix we give some complementary accounts of selected topics on ∞-categories.

B.1. **Kan model structure.** Let us explain the model structure on the category Set_Δ of simplicial sets which is called the Kan model structure in [Lur1, §A.2.7]. We begin with

Definition B.1.1. A simplicial map $f : X \to Y$ is called a *Kan fibration* if f has the right lifting property with respect to all horn inclusions $\Lambda^n_i \hookrightarrow \Delta^n$, i.e., if given any diagram

$$\begin{array}{ccc} \Lambda^n_i & \longrightarrow & X \\ \downarrow & & \downarrow \\ \Delta^n & \longrightarrow & Y \end{array}$$

with arbitrary $n \in \mathbb{N}$ and any $i = 0, 1, \ldots, n$, there exists a simplicial map $\Delta^n \to X$ making the diagram commutative.

Next, recall the classical theory telling that there exists an adjoint pair

$$|-| : \mathrm{Set}_\Delta \rightleftarrows \mathcal{CG} : \mathrm{Sing} \tag{B.1.1}$$

of functors where $\mathcal{CG}$ denotes the category of compactly generated Hausdorff topological spaces. The functor $|-|$ is called the *geometric realization*.

Now we introduce

Fact B.1.2 ([GJ, Chap. 1 §11])**.** The following data gives Set_Δ a model structure.

- A simplicial map $f : X \to Y$ is a cofibration if it is a monomorphism, i.e., the map $X_n \to Y_n$ is injective for each $n \in \mathbb{N}$.
- A simplicial map is a fibration if it is a Kan fibration.
- A simplicial map $f : X \to Y$ is a weak equivalence if the induced map $|X| \to |Y|$ of geometric realizations is a homotopy equivalence of topological spaces.

The obtained model structure is called the *Kan model structure* on Set_Δ.

B.2. **Homotopy category of an ∞-category.** In Definition 1.3.1, we denoted by $\mathcal{CG}$ the category of compactly generated weakly Hausdorff topological spaces. The classical theory tells us that for each $X \in \mathcal{CG}$ there exists a CW complex $X' \in \mathcal{CW}$, giving rise to a well-defined functor

$$\theta : \mathcal{CG} \longrightarrow \mathcal{H}, \quad X \longmapsto [X] := X'.$$

We call $[X] \in \mathcal{H}$ the *homotopy type* of X. Now we can define the homotopy category of a topological category.

Definition ([Lur1, pp. 16–17])**.** For a topological category $\mathcal{C}$, we denote by $\mathrm{h}\,\mathcal{C}$ the category enriched over $\mathcal{H}$ defined as follows, and call it the *homotopy category* of $\mathcal{C}$.

- The objects of $\mathrm{h}\,\mathcal{C}$ are defined to be the objects of $\mathcal{C}$.
- For $X, Y \in \mathcal{C}$, we set $\mathrm{Map}_{\mathrm{h}\,\mathcal{C}}(X, Y) := [\mathrm{Map}_{\mathcal{C}}(X, Y)] \in \mathcal{H}$.
- Composition of morphisms in $\mathrm{h}\,\mathcal{C}$ is given by the application of θ to composition of morphisms in $\mathcal{C}$.

Next we explain the homotopy category of a simplicial category. Recall the adjunction

$$|-| : \mathrm{Set}_\Delta \rightleftarrows \mathcal{CG} : \mathrm{Sing}$$

in (B.1.1). Composing $|-| : \mathrm{Set}_\Delta \to \mathcal{CG}$ with $\theta : \mathcal{CG} \to \mathcal{H}$, we have a functor

$$[\cdot] : \mathrm{Set}_\Delta \longrightarrow \mathcal{H}, \quad S \longmapsto [S] := \theta(|S|).$$

Definition B.2.1. For a simplicial set $S \in \mathrm{Set}_\Delta$, we call $[S]$ the *homotopy type* of S.

Recall the category Cat_Δ of simplicial categories (Definition 1.3.3). Applying this functor $[\cdot] : \mathrm{Set}_\Delta \to \mathcal{H}$ to the simplicial sets of morphisms, we obtain another functor

$$\mathrm{h} : \mathrm{Cat}_\Delta \longrightarrow (\text{categories enriched over } \mathcal{H}), \quad \mathrm{C} \longmapsto \mathrm{h}\,\mathrm{C}.$$

Definition B.2.2 ([Lur1, p. 19])**.** For a simplicial category $\mathfrak{C} \in \mathrm{Cat}_\Delta$, we call $\mathrm{h}\,\mathfrak{C}$ in the above construction the *homotopy category* of $\mathfrak{C}$.

Now we explain the homotopy category of an ∞-category. Recall that we denote by Cat_Δ the category of simplicial categories (Definition 1.3.3). Then we can define a functor

$$\mathfrak{C}[-] : \mathrm{Set}_\Delta \longrightarrow \mathrm{Cat}_\Delta$$

as follows [Lur1, §1.1.5]: For a finite non-empty linearly ordered set J, we construct a simplicial category $\mathfrak{C}[\Delta^J]$ as follows: The objects of $\mathfrak{C}[\Delta^J]$ are the elements of J. For $i, j \in J$ with $i \le j$, the simplicial set $\mathrm{Map}_{\mathfrak{C}[\Delta^J]}(i, j)$ is given by the nerve of the poset $\{I \subset J \mid i, j \in I,\ \forall k \in I\ i \le k \le j\}$. For $i > j$ we set $\mathrm{Map}_{\mathfrak{C}[\Delta^J]}(i, j) := \emptyset$. The resulting functor $\mathfrak{C} : \Delta \to \mathrm{Cat}_\Delta$ extends uniquely to a functor $\mathfrak{C}[-] : \mathrm{Set}_\Delta \to \mathrm{Cat}_\Delta$, and we denote by $\mathfrak{C}[S]$ the image of $S \in \mathrm{Set}_\Delta$.

Definition B.2.3 ([Lur1, Definition 1.1.5.14])**.** Let S be a simplicial set. The *homotopy category* $\mathrm{h}\,S$ of S is defined to be

$$\mathrm{h}\,S := \mathrm{h}\,\mathfrak{C}[S],$$

the homotopy category of the simplicial category $\mathfrak{C}[S]$.

The *homotopy category* of an ∞-category C is defined to be the homotopy category $\mathrm{h}\,\mathsf{C}$ of C as a simplicial set.

Noting that the homotopy category $\mathrm{h}\,S$ is enriched over $\mathcal{H}$, we denote by $\mathrm{Hom}_{\mathrm{h}\,S}(-,-) \in \mathcal{H}$ its Hom space. Then we can introduce

Definition B.2.4 ([Lur1, Definition 1.2.2.1])**.** For a simplicial set S and its vertices $x, y \in S$, we define

$$\mathrm{Map}_S(x, y) := \mathrm{Hom}_{\mathrm{h}\,S}(x, y) \in \mathcal{H}$$

and call it the *mapping space* from x to y in S.

For use in the main text, we recall the construction of Kan complexes which represent mapping spaces.

Definition B.2.5. For a simplicial set S and vertices $x, y \in S$, we defined a simplicial set $\mathrm{Hom}^{\mathrm{R}}_{\mathrm{S}}(x, y)$ by

$$\mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^n, \mathrm{Hom}^{\mathrm{R}}_{\mathrm{S}}(x, y)) =$$
$$\{z : \Delta^{n+1} \to S \mid \text{simplicial maps},\ z|_{\Delta^{\{n+1\}}} = y,\ z|_{\Delta^{\{0,\ldots,n\}}} \text{ is a constant complex at the vertex } x\}.$$

The face and degeneracy maps are induced by those on S_{n+1}.

If C is an ∞-category, then $\mathrm{Hom}^{\mathrm{R}}_{\mathsf{C}}(x, y)$ is a Kan complex by [Lur1, Proposition 1.2.2.3].

By [Lur1, Proposition 2.2.4.1], we have an adjunction

$$|-|_{Q^\bullet} : \mathrm{Set}_\Delta \rightleftarrows \mathrm{Set}_\Delta : \mathrm{Sing}_{Q^\bullet}$$

which gives a Quillen autoequivalence on the category Set_Δ equipped with the Kan model structure. Now we have

Fact ([Lur1, Proposition 2.2.4.1])**.** For an ∞-category C and objects $x, y \in \mathsf{C}$, there is a natural equivalence of simplicial sets

$$\left|\mathrm{Hom}^{\mathrm{R}}_{\mathsf{C}}(x, y)\right|_{Q^\bullet} \longrightarrow \mathrm{Map}_{\mathfrak{C}[S]}(x, y).$$

Here $\mathfrak{C} : \mathrm{Set}_\Delta \to \mathrm{Cat}_\Delta$ is the functor explained at (1.3.1).

B.3. **Over-∞-categories and under-∞-categories.** This subsection is based on [Lur1, §1.2.9].

For simplicial sets S and S', their *join* [Lur1, Definition 1.2.8.1] is denoted by $S \star S'$. The join of ∞-categories is an ∞-category [Lur1, Proposition 1.2.8.3].

Definition B.3.1 ([Lur1, §1.2.9])**.** Let $p : K \to \mathsf{C}$ be a simplicial map from a simplicial set K to an ∞-category C.

(1) Consider the simplicial set $\mathsf{C}_{/p}$ defined by
$$(\mathsf{C}_{/p})_n := \mathrm{Hom}_{/p}(\Delta^n \star K, \mathsf{C}).$$
The subscript $/p$ in the right hand side means that we only consider those morphisms $f : S \star K \to \mathsf{C}$ such that $f|_K = p$. Then $\mathsf{C}_{/p}$ is an ∞-category, and called the *over-∞-category of objects over* p

(2) For $X \in \mathsf{C}$, we denote by $\mathsf{C}_{/X}$ the over-∞-category $\mathsf{C}_{/p}$ where $p : \Delta^0 \to \mathsf{C}$ has X as its image.

(3) Dually, the *under-∞-category* $\mathsf{C}_{p/}$ is the ∞-category defined by $(\mathsf{C}_{p/})_n := \mathrm{Hom}_{p/}(K \star \Delta^n, \mathsf{C})$, where the subscript $p/$ means that we only consider those morphisms $f : K \star S \to \mathsf{C}$ such that $f|_K = p$.

(4) For $X \in \mathsf{C}$, we denote by $\mathsf{C}_{X/}$ the under-∞-category $\mathsf{C}_{p/}$ where $p : \Delta^0 \to \mathsf{C}$ has X as its image.

The over-∞-category $\mathsf{C}_{/p}$ is characterized by the universal property

$$\mathrm{Hom}_{\mathrm{Set}_\Delta}(S, \mathsf{C}_{/p}) = \mathrm{Hom}_{/p}(S \star K, \mathsf{C})$$

for any simplicial set S. We can characterize an under-∞-category by a similar universal property.

Note that for an injective simplicial map $j : L \to K$ we have a natural functor $\mathsf{C}_{/p} \to \mathsf{C}_{/p \circ j}$.

By this universality one can deduce the following consequences. We omit the proof.

Corollary B.3.2. Let C be an ∞-category.

(1) For any $X \in \mathsf{C}$ there exists a functor
$$\mathsf{C}_{/X} \longrightarrow \mathsf{C}$$
of ∞-categories. It will be called the *canonical functor* of $\mathsf{C}_{/X}$.

(2) For a morphism $f : X \to Y$ in C, we have a functor $\mathsf{C}_{/X} \to \mathsf{C}_{/Y}$ of ∞-categories induced by composition with f. We also have functors $\mathsf{C}_{/f} \to \mathsf{C}_{/X}$ and $\mathsf{C}_{/f} \to \mathsf{C}_{/Y}$, where $\mathsf{C}_{/f}$ is defined to be $\mathsf{C}_{/p}$ with $p : \Delta^1 \to \mathsf{C}$ representing f. These functors form a commutative triangle
$$\begin{array}{ccc} & \mathsf{C}_{/f} & \\ \swarrow & & \searrow \\ \mathsf{C}_{/X} & \longrightarrow & \mathsf{C}_{/Y} \end{array}$$
in the ∞-category Cat_∞ of ∞-categories (Definition 1.6.1).

Remark. In [Lur1] and [J] the canonical functor is called the projection, but we avoid this terminology (see Definition 3.6.3 for the reason).

Definition B.3.1 works for any simplicial set S instead for an ∞-category S. The statement that $\mathsf{C}_{/p}$ for an ∞-category C is indeed an ∞-category is shown in [Lur1, Proposition 2.1.2.2].

B.4. **Limits and colimits in ∞-categories.** The limits and colimits in ∞-categories are defined in terms of final and initial objects.

Definition ([Lur1, Definition 1.2.13.4])**.** Let C be an ∞-category, K be a simplicial set and $p : K \to \mathsf{C}$ be a simplicial map.

(1) A *colimit* of p is an initial object of the under-∞-category $\mathsf{C}_{p/}$, and denoted by $\varinjlim p$.
(2) A *limit* of p is a final object of the over-∞-category $\mathsf{C}_{/p}$, and denoted by $\varprojlim p$.

Remark B.4.1. A (co)limit of p is not unique if it exists, but by [Lur1, Proposition 1.2.12.9] the full sub-∞-category of (co)limits is either empty or is a contractible Kan complex.

We have the following restatement of (co)limit.

Fact ([Lur1, Remark 1.2.13.5])**.** A colimit of $p : K \to \mathsf{C}$ can be identified with a simplicial map $\overline{p} : K^{\triangleright} \to \mathsf{C}$ extending p. Similarly, a limit of p is identified with a simplicial map $\overline{p} : K^{\triangleleft} \to \mathsf{C}$ extending p.

Here we used

Definition B.4.2 ([Lur1, Notation 1.2.8.4])**.** For a simplicial set K, we denote by $K^{\triangleright} := K \star \Delta^0$ the *right cone* K, where $\star$ denotes the join of simplicial sets. We also denote by $K^{\triangleleft} := \Delta^0 \star K$ the *left cone* of K.

Remark B.4.3. As explained in [Lur1, §4.2.4], colimits in the ∞-category are compatible with homotopy colimits in simplicial categories, and limits are compatible with the homotopy limits.

Let us explain a few examples of limits and colimits in ∞-categories, following [Lur1, §4.4].

- We regard a set A as a category by $\mathrm{Hom}_A(i, i) = *$ for $i \in A$ and $\mathrm{Hom}_A(i, j) = \emptyset$ for $i \neq j$. We further consider A as the simplicial set which is the nerve of this category.

  Definition B.4.4 ([Lur1, §4.4.1])**.** Let A be a set, C an ∞-category and $p : A \to \mathsf{C}$ be a map. Thus p is identified with the family $\{X_a \mid a \in A\}$ of objects in C. Then a colimit $\varinjlim p$ is called a *coproduct* of $\{X_a \mid a \in A\}$, and denoted by $\coprod_{a \in A} X_a$. Dually, a limit $\varprojlim p$ is called a *product* of $\{X_a \mid a \in A\}$, and denoted by $\prod_{a \in A} X_a$.

  Under Remark B.4.3, the corresponding object in a simplicial category is the homotopy coproduct.
- Let C be an ∞-category. A simplicial map $\Delta^1 \times \Delta^1 \to \mathsf{C}$ is called a *square* in C. It will be typically depicted as

(B.4.1)
$$\begin{array}{ccc} X' & \xrightarrow{q'} & X \\ {\scriptstyle p'}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ Y' & \xrightarrow[q]{} & Y \end{array}$$

  Since there are isomorphisms $(\Lambda^2_0)^{\triangleright} \simeq \Delta^1 \times \Delta^1 \simeq (\Lambda^2_2)^{\triangleleft}$ of simplicial sets, we can introduce

  Definition B.4.5 ([Lur1, §4.4.2])**.** Let $\sigma : \Delta^1 \times \Delta^1 \to \mathsf{C}$ be a square in an ∞-category C.
  (1) If σ is a limit of $\sigma|_{\Lambda^2_2}$ viewing $\Delta^1 \times \Delta^1 \simeq (\Lambda^2_2)^{\triangleleft}$, then it is called a *pullback square* or a *cartesian square*. If the square (B.4.1) is a pull-back square, then we write $X' = X \times_{p,Y,q} Y'$ or simply $X' = X \times_Y Y'$, and call X' a *pullback* or a *base change* or a *fiber product*.
  (2) If σ is a colimit of $\sigma|_{\Lambda^2_0}$ viewing $\Delta^1 \times \Delta^1 \simeq (\Lambda^2_0)^{\triangleright}$, then it is called a *pushout square* or a *cocartesian square*, and we write $Y = X \coprod^{X'} Y'$ and call Y a *pushout*.

  Under Remark B.4.3, the corresponding objects in a simplicial category are the homotopy pullback and the homotopy pushout.
- Finally let D be the category depicted by the diagram

$$X \underset{G}{\overset{F}{\rightrightarrows}} Y$$

  Definition B.4.6. Let $p : \mathsf{N}(\mathrm{D}) \to \mathsf{C}$ be a simplicial map from the nerve of D (Definition 1.2.1) to an ∞-category C. Set $f := p(F)$ and $g := p(G)$. A colimit of p is called a *coequalizer* of f and g.

B.5. **Adjunctions.** We now introduce the notion of adjoint functors of ∞-categories, for which we prepare several definitions on simplicial maps.

Definition. Let $f : X \to S$ be a simplicial map.

(1) f is called a *trivial fibration* if f has the right lifting property with respect to all inclusions $\partial\Delta^n \hookrightarrow \Delta^n$.
(2) f is called a *trivial Kan fibration* if it is a Kan fibration (Definition B.1.1) and a trivial fibration.
(3) f is called an *inner fibration* if f has the right lifting property with respect to horn inclusions $\Lambda^n_i \hookrightarrow \Delta^n$ with $0 < i < n$.

Definition ([Lur1, Definition 2.4.1.1])**.** Let $p : X \to S$ be an inner fibration of simplicial sets. Let $f : x \to y$ be an edge in X. f is called *p-cartesian* if the induced map $X_{/f} \to X_{/y} \times_{S_{/p(y)}} S_{/p(f)}$ is a trivial Kan fibration.

Definition B.5.1 ([Lur1, Definition 2.4.2.1])**.** Let $p : X \to S$ be a simplicial map.

(1) p is called a *cartesian fibration* if the following conditions are satisfied.
  - The map p is an inner fibration.
  - For every edge $f : x \to y$ of S and every vertex $\widetilde{y}$ of X with $p(\widetilde{y}) = y$, there exists a p-cartesian edge $\widetilde{f} : \widetilde{x} \to \widetilde{y}$ with $p(\widetilde{f}) = f$.
(2) p is called a *cocartesian fibration* if the opposite map $p^{\mathrm{op}} : X^{\mathrm{op}} \to S^{\mathrm{op}}$ is a cartesian fibration.

Now we explain the main definition in this subsection.

Definition B.5.2 ([Lur1, Definition 5.2.2.1])**.** Let B and C be ∞-categories. An *adjunction* between B and C is a simplicial map $p : M \to \Delta^1$ which is both a cartesian fibration and a cocartesian fibration together with equivalences $f : \mathsf{B} \to p^{-1}\{0\}$ and $g : \mathsf{C} \to p^{-1}\{1\}$. In this case we say that f is *left adjoint* to g and that g is *right adjoint* to f, and denote

$$f : \mathsf{B} \rightleftarrows \mathsf{C} : g.$$

As in the ordinary category theory, an adjunction can be restated by a unit (and by a counit).

Definition B.5.3 ([Lur1, Definition 5.2.2.7])**.** Let us given a pair of functors $(f : \mathsf{B} \to \mathsf{C}, g : \mathsf{C} \to \mathsf{B})$ of ∞-categories. A *unit transformation* for (f, g) is a morphism $u : \mathrm{id}_{\mathsf{B}} \to g \circ f$ in $\mathsf{Fun}(\mathsf{B}, \mathsf{B})$ such that for any $B \in \mathsf{B}$ and $C \in \mathsf{C}$ the composition

$$\mathrm{Map}_{\mathsf{C}}(f(B), C) \longrightarrow \mathrm{Map}_{\mathsf{B}}(g(f(B)), g(C)) \xrightarrow{u(C)} \mathrm{Map}_{\mathsf{B}}(B, g(C))$$

is an isomorphism in the homotopy category $\mathcal{H}$ of spaces.

Dually we have the notion of a *counit transformation* $c : f \circ g \to \mathrm{id}_{\mathsf{B}}$.

Fact B.5.4 ([Lur1, Proposition 5.2.2.8])**.** Let $(f : \mathsf{B} \to \mathsf{C}, g : \mathsf{C} \to \mathsf{B})$ be a pair of functors of ∞-categories. Then the following conditions are equivalent.

- The functor f is a left adjoint to g.
- There exists a unit transformation $u : \mathrm{id}_{\mathsf{B}} \to g \circ f$.

We have a dual statement for right adjoint and counit transformation.

For an existence criterion of an adjoint functor via exactness, see Fact B.10.3.

In the main text we will use following statement repeatedly.

Fact ([Lur1, Proposition 5.2.2.6])**.** Let $f_i : \mathsf{C}_i \to \mathsf{C}_{i+1}$ $(i = 1, 2)$ be functors of ∞-categories. Suppose that f_i has a right adjoint g_i $(i = 1, 2)$. Then $g_2 \circ g_1$ is right adjoint to $f_2 \circ f_1$.

B.6. **The underlying ∞-category of a simplicial model category.** In the main text we often translate the model-categorical arguments in [TVe1, TVe2] into the ∞-categorical arguments. Such a translation is possible by the notion of the *underlying ∞-category*, which is explained here.

Let us recall the monoidal model category structure [Lur1, Definition A.3.1.2] on the category Set_{Δ} of simplicial sets given by the cartesian product and the Kan model structure (Fact B.1.2). Here the cartesian product of $S, T \in \mathrm{Set}_{\Delta}$ is given by $(S \times T)_n := S_n \times T_n$, where the latter $\times$ means the cartesian product in the category Set.

Definition B.6.1 ([Lur1, Definition A.3.1.5], [H, Definition 4.2.18])**.** A *simplicial model category* $\mathfrak{A}$ is a simplicial category equipped with a model structure satisfying the following conditions.

- The category $\mathfrak{A}$ is tensored and cotensored over the monoidal model category Set_{Δ} in the sense of [Lur1, Remark A.1.4.4].
- The action map $\otimes : \mathfrak{A} \times \mathrm{Set}_{\Delta} \to \mathfrak{A}$ arising from the tensored structure is a left Quillen bifunctor.

One can construct an ∞-category from a simplicial model category $\mathfrak{A}$. Let us denote by $\mathfrak{A}^\circ \subset \mathfrak{A}$ the full subcategory of fibrant-cofibrant objects, which is a fibrant simplicial category. Taking the simplicial nerve (Definition 1.4.1), we obtain an ∞-category $\mathsf{N}_{\mathrm{spl}}(\mathfrak{A}^\circ)$ by Fact 1.4.2.

Definition B.6.2 ([Lur1, §A.2]). We call the ∞-category $\mathsf{N}_{\mathrm{spl}}(\mathfrak{A}^\circ)$ the *underlying ∞-category* of the simplicial model category $\mathfrak{A}$.

Let us cite a result on adjunctions.

Fact B.6.3 ([Lur1, Proposition 5.2.4.6]). Given a Quillen adjunction $\mathfrak{A} \rightleftarrows \mathfrak{A}'$ of simplicial model categories, there is a natural adjunction $\mathsf{N}_{\mathrm{spl}}(\mathfrak{A}^\circ) \rightleftarrows \mathsf{N}_{\mathrm{spl}}(\mathfrak{A}')$ of the underlying ∞-categories.

B.7. **∞-localization.** We cite from [Lur2, §1.3.4] terminologies on localization of ∞-categories.

Definition B.7.1 ([Lur2, Definition 1.3.4.1]). Let C be an ∞-category and W be a collection of morphisms in C. We say that *a functor $f : \mathsf{C} \to \mathsf{D}$ exhibits D as the ∞-category obtained from C by inverting the set of morphisms W* if, for every ∞-category B, composition with f induces a fully faithful embedding $\mathsf{Fun}(\mathsf{D}, \mathsf{B}) \to \mathsf{Fun}(\mathsf{C}, \mathsf{B})$, whose image is the collection of functors $F : \mathsf{C} \to \mathsf{B}$ mapping each morphism in W to an equivalence in B. In this case we denote $\mathsf{C}[W^{-1}] := \mathsf{D}$.

Note that $\mathsf{C}[W^{-1}]$ is determined uniquely up to equivalence by C and W, Note also that $\mathsf{C}[W^{-1}]$ exists for any C and W. See [Lur2, Remark 1.3.4.2] for an account.

Now let us recall

Definition B.7.2 ([Lur1, Definition 5.2.7.2]). A functor $L : \mathsf{C} \to \mathsf{D}$ of ∞-categories is called a *localization* (*functor*) if L has a fully faithful right adjoint.

Then by [Lur1, Proposition 5.2.7.12] we have

Fact B.7.3 ([Lur2, Example 1.3.4.3]). Let C be an ∞-category and $L : \mathsf{C} \to \mathsf{C}_0$ be a localization. We denote by $i : \mathsf{C}_0 \hookrightarrow \mathsf{C}$ the fully faithful right adjoint of L. Define W to be the collection of those morphisms α in C such that $L(\alpha)$ is an equivalence in C_0. Then the composite $\mathsf{C}_0 \xrightarrow{i} \mathsf{C} \xrightarrow{L} \mathsf{C}[W^{-1}]$ is an equivalence of ∞-categories.

B.8. **Presentable ∞-categories.** Most of the ∞-categories appearing in the main text are presentable in the sense of [Lur1, §5.5], and enjoy a good property with respect to taking limits. The notion of a presentable ∞-category is an ∞-theoretic analogue of the notion of a locally presentable category.

First we want to introduce the notion of a filtered ∞-category, which is an ∞-theoretic analogue of filtered categories. Recall the notation $K^{\triangleright}$ of the right cone of a simplicial set K (Definition B.4.2).

Definition B.8.1. Let κ be a regular cardinal. An ∞-category C is called *κ-filtered* if for any κ-small simplicial set K and any simplicial map $f : K \to \mathsf{C}$ there exists a simplicial map $\overline{f} : K^{\triangleright} \to \mathsf{C}$ extending f.

Next we turn to the definition of ind-objects in an ∞-category.

Definition ([Lur1, Definition 5.3.1.7]). For an ∞-category C and a regular cardinal κ, we denote by $\mathrm{Ind}_\kappa(\mathsf{C})$ the full sub-∞-category of $\mathsf{PSh}(\mathsf{C})$ spanned by the functors $f : \mathsf{C}^{\mathrm{op}} \to \mathcal{S}$ classifying right fibrations $\widetilde{\mathsf{C}} \to \mathsf{C}$, where the ∞-category $\widetilde{\mathsf{C}}$ is κ-filtered. An object of $\mathrm{Ind}_\kappa(\mathsf{C})$ is called an *ind-object of* C.

Here we used

Definition ([Lur1, Definition 2.0.0.3]). A simplicial map $f : X \to S$ is called a *right fibration* if f has the right lifting property with respect to all the horn inclusions $\Lambda^n_i \hookrightarrow \Delta^n$ for any $0 < i \le n$.

Finally we introduce

Definition ([Lur1, Definition 5.4.2.1]). Let C be an ∞-category.

(1) Let κ be a regular cardinal. We call C *κ-accessible* if there exists a small ∞-category C^0 and an equivalence $\mathrm{Ind}_\kappa(\mathsf{C}^0) \to \mathsf{C}$.
(2) C is called *accessible* if it is κ-accessible for some regular cardinal κ.

Accessible ∞-categories have a nice property in terms of compact objects. In order to state its precise definition, for an ∞-category C and an object $C \in \mathsf{C}$ let us denote by $j_C : C \to \widehat{\mathcal{S}}$ the composition

$$\mathsf{PSh}(\mathsf{C}) = \mathsf{Fun}(\mathsf{C}^{\mathrm{op}}, \widehat{\mathcal{S}}) \longrightarrow \mathsf{Fun}(\{s\}, \widehat{\mathcal{S}}) \xrightarrow{\sim} \widehat{\mathcal{S}}$$

and call it the *functor corepresented by C*. Here $\widehat{\mathcal{S}}$ denotes the ∞-category of (not necessarily small) spaces (recall that $\mathcal{S}$ denotes the ∞-category of small spaces).

Definition B.8.2 ([Lur1, Definition 5.3.4.5]). Let κ be a regular cardinal, and C be an ∞-category admitting small κ-filtered colimits.

(1) A functor $f : \mathsf{C} \to \mathsf{D}$ of ∞-categories is called *κ-continuous* if it preserves κ-filtered colimits.
(2) Assume C admits κ-filtered colimits. Then an object $C \in \mathsf{C}$ is called *κ-compact* if the functor $j_C : C \to \widehat{\mathsf{S}}$ corepresented by C is κ-continuous.

Definition B.8.3 ([Lur1, Definition 5.5.0.1]). An ∞-category C is called *presentable* if it is accessible and admits arbitrary small colimits.

Let us cite an equivalent definition of presentable ∞-categories. For that we need

Definition. Let C be an ∞-category.

(1) C is *essentially small* if it is κ-compact as an object of Cat_∞ for some small regular cardinal κ.
(2) C is *locally small* if for any objects $X, Y \in \mathsf{C}$ the mapping space $\operatorname{Map}_{\mathsf{C}}(X, Y)$ is essentially small as an ∞-category.

See [Lur1, Proposition 5.4.1.2] for equivalent definitions of essential smallness.

Fact B.8.4 ([Lur1, Theorem 5.5.1.1]). For an ∞-category C, the following conditions are equivalent.

(1) C is presentable.
(2) C is locally small and admits small colimits, and there exists a regular cardinal κ and a small set S of κ-compact objects of C such that every object of C is a colimit of a small diagram taking values in the full sub-∞-category of C spanned by S.

Presentable categories enjoy much nice properties as explained in [Lur1, §5.5].

Fact B.8.5. Let C be a presentable ∞-category.

(1) C admits arbitrary small limits [Lur1, Corollary 5.5.2.4].
(2) The product of presentable categories is presentable [Lur1, Proposition 5.5.3.5].
(3) The functor ∞-category of presentable categories is presentable [Lur1, Proposition 5.5.3.6].
(4) The over-∞-category $\mathsf{C}_{/p}$ and under-∞-category $\mathsf{C}_{p/}$ with respect to a simplicial morphism p of small ∞-categories are presentable [Lur1, Proposition 5.5.3.10, Proposition 5.5.3.11].

B.9. **Truncation functor.** We close this subsection by explaining the truncation functor for an ∞-category. We begin with the truncation of objects.

Definition B.9.1 ([Lur1, Definition 5.5.6.1]). Let C be an ∞-category.

(1) Let $k \in \mathbb{Z}_{\geq -1}$. An object $C \in \mathsf{C}$ is called *k-truncated* if for any $D \in \mathsf{C}$ the space $\operatorname{Map}_{\mathsf{C}}(D, C)$ is k-truncated, i.e., $\pi_i \operatorname{Map}_{\mathsf{C}}(D, C) = *$ for $i \in \mathbb{Z}_{\geq k+1}$.
(2) A *discrete* object is defined to be a 0-truncated object.
(3) An object C is called *(-2)-truncated* if it is a final object of C (Definition 1.3.7).
(4) For $k \in \mathbb{Z}_{\geq -2}$, we denote by $\tau_{\leq k}\mathsf{C}$ the full sub-∞-category of C spanned by k-truncated objects.

The truncation of morphisms is given by

Definition B.9.2 ([Lur1, Definition 5.5.6.8]).

(1) Let $k \in \mathbb{Z}_{\geq -2}$. A simplicial map $f : X \to Y$ of Kan complexes is *k-truncated* if the homotopy fibers of f taken over any base point of Y are k-truncated.
(2) A morphism $f : C \to D$ in an ∞-category C is *k-truncated* if for any $E \in \mathsf{C}$ the simplicial map $\operatorname{Map}_{\mathsf{C}}(E, C) \to \operatorname{Map}_{\mathsf{C}}(E, D)$ given by the composition with f is k-truncated in the sense of (1).

Now we have

Fact ([Lur1, Proposition 5.5.6.18]). For a presentable ∞-category C and $k \in \mathbb{Z}_{\geq -2}$, the inclusion $\tau_{\leq k}\mathsf{C} \hookrightarrow \mathsf{C}$ has an accessible left adjoint.

In the last line we used

Definition B.9.3 ([Lur1, Definition 5.4.2.5]). Let C be an accessible ∞-category. A functor $F : \mathsf{C} \to \mathsf{C}'$ is called *accessible* if it is κ-continuous (Definition B.8.2) for some regular cardinal κ.

Now we may introduce

Definition B.9.4. For a presentable ∞-category C and $k \in \mathbb{Z}_{\geq -2}$, a left adjoint to the inclusion $\tau_{\leq k}\mathsf{C} \hookrightarrow \mathsf{C}$ is denoted by

$$\tau_{\leq k} : \mathsf{C} \longrightarrow \tau_{\leq k}\mathsf{C}$$

and called the *truncation functor*.

Here we used "the" since it is unique up to contractible ambiguity [Lur1, Remark 5.5.6.20]. It is obviously a localization functor (Definition B.7.2).

Another usage of truncation is

Definition B.9.5. A morphism in an ∞-category is a *monomorphism* if it is (-1)-truncated (Definition B.9.2).

Let us cite a redefinition of monomorphisms.

Fact B.9.6. A morphism $f : X \to Y$ in an ∞-category C is a monomorphism if and only if the functor $\mathsf{C}_{/f} \to \mathsf{C}_{/Y}$ is fully faithful (Definition 1.3.4).

For the use in the main text, let us also cite

Fact B.9.7 ([Lur1, Lemma 5.5.6.15])**.** Let C be an ∞-category which admits finite limits. Then a morphism $f : X \to Y$ in C is a monomorphism if and only if the diagonal $X \to X \times_{f,Y,f} X$ is an isomorphism.

Strictly speaking, this statement is a special case $k = -1$ in loc. cit.

B.10. **Exact functors of ∞-categories.** We close this section by recalling left and right exact functors of ∞-categories.

Definition B.10.1 ([Lur1, Definition 2.0.0.3, Definition 5.3.2.1])**.**
(1) A simplicial map $f : X \to S$ is a *left fibration* (resp. *right fibration*) if f has the right lifting property (resp. *left lifting property*) with respect to all the horn inclusions $\Lambda^n_i \hookrightarrow \Delta^n$ for any $0 \le i < n$.
(2) Let $F : \mathsf{B} \to \mathsf{C}$ be a functor of ∞-categories and κ be a regular cardinal κ, F is *κ-left exact* (resp. *κ-right exact*) if for any left fibration (resp. right fibration) $\mathsf{C}' \to \mathsf{C}$ where C' is κ-filtered (Definition B.8.1), the ∞-category $\mathsf{B}' = \mathsf{B} \times_{\mathsf{C}} \mathsf{C}'$ is also κ-filtered.
(3) A functor of ∞-categories is *left exact* (resp. *right exact*) if it is ω-left exact (resp. ω-right exact), where ω denotes the lowest transfinite ordinal number.
(4) A functor of ∞-category is *exact* if it is both left exact and right exact.

We will repeatedly use the following criterion of exactness in the main text.

Fact B.10.2 ([Lur1, Proposition 5.3.2.9])**.** Let $F : \mathsf{B} \to \mathsf{C}$ be a functor of ∞-categories and κ be a regular cardinal.
(1) If f is κ-left exact, then F preserves all κ-small colimits which exists in B.
(2) If B admits κ-small limits and F preserves κ-small colimits, then F is κ-right exact.
Dual statements hold for right-exactness.

Recall the notion of adjoint functors (Definition B.5.2). Now we have the following criterion on existence of adjunctions.

Fact B.10.3 ([Lur1, Corollary 5.5.2.9])**.** Let $F : \mathsf{B} \to \mathsf{C}$ be a functor of presentable ∞-categories.
(1) F has a right adjoint if and only if it is right exact.
(2) F has a left adjoint if and only if it is left exact.

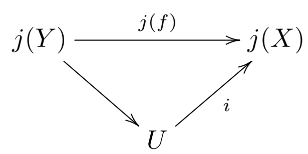
Appendix C. ∞-topoi

In this section we give some complementary accounts on ∞-topoi.

C.1. **Sheaves on ∞-sites.** Let C be an ∞-category. Recall the Yoneda embedding $j : \mathsf{C} \to \mathsf{PSh}(\mathsf{C})$ in Definition 1.5.2. By [Lur1, Proposition 6.2.2.5], for each $X \in \mathsf{C}$ we have a bijection

$$\sigma_X : \mathrm{Sub}(j(X)) \xrightarrow{\sim} \mathrm{Sieve}(X), \quad (i : U \hookrightarrow j(X)) \longmapsto \mathsf{C}_{/X}(U) \tag{C.1.1}$$

between the set $\mathrm{Sub}(j(X))$ of monomorphisms $U \hookrightarrow j(X)$ (Definition B.9.5) and the set $\mathrm{Sieve}(X)$ of all sieves on X. Here $\mathsf{C}_{/X}(U)$ denotes the full sub-∞-category of C spanned by those objects $f : Y \to X$ of $\mathsf{C}_{/X}$ such that there exists a commutative triangle



Now we recall

Definition C.1.1 ([Lur1, Definition 5.5.4.2]). For an ∞-category B and a set S of morphisms in B, we call $Z \in \mathsf{C}$ to be *S-local* if for any $(s : X \to Y) \in S$ the composition with s induces an isomorphism $\mathrm{Map}_{\mathsf{C}}(Y, Z) \xrightarrow{\sim} \mathrm{Map}_{\mathsf{C}}(X, Z)$ in the homotopy category $\mathrm{h}\,\mathcal{S}$ of spaces.

Then we can introduce

Definition C.1.2. Let (C, τ) be an ∞-site.

(1) Define the set S_τ by

$$S_\tau := \bigcup_{X \in \mathsf{C}} \sigma_X^{-1}(\mathrm{Cov}_\tau(X)),$$

the set of all monomorphisms $U \to j(X)$ corresponding to covering sieves $\mathsf{C}^{(0)}_{/X}$ of τ under the bijection σ_X (C.1.1). A presheaf $\mathcal{F} \in \mathsf{PSh}(\mathsf{C})$ is called a *τ-sheaf* if it is S_τ-local.

(2) We denote by

$$\mathsf{Sh}(\mathsf{C}, \tau) \subset \mathsf{PSh}(\mathsf{C})$$

the full sub-∞-category spanned by τ-sheaves.

As explained in Fact 1.7.4 the ∞-category $\mathsf{Sh}(\mathsf{C}, \tau)$ is always an ∞-topos.

C.2. **Yoneda embedding of ∞-topoi.** Let (C, τ) be an ∞-site and $L : \mathsf{PSh}(\mathsf{C}) \to \mathsf{Sh}(\mathsf{C}, \tau)$ be a left adjoint of the inclusion $\mathsf{Sh}(\mathsf{C}, \tau) \hookrightarrow \mathsf{PSh}(\mathsf{C})$ of the ∞-topos $\mathsf{Sh}(\mathsf{C}, \tau)$. Also let $j : \mathsf{C} \hookrightarrow \mathsf{PSh}(\mathsf{C})$ be the Yoneda embedding (Definition 1.5.2) We first recall

Fact ([Lur1, Lemma 6.2.2.16]). Let (C, τ), L and j as above, and let $i : U \to j(X)$ be a monomorphism in $\mathsf{PSh}(\mathsf{C})$ corresponding to a sieve $\mathsf{C}^{(0)}_{/X}$ on $X \in \mathsf{C}$ under the bijection (C.1.1). Then $L \circ i$ is an equivalence if and only if $\mathsf{C}^{(0)}_{/X}$ is a covering sieve.

This fact implies that given an ∞-site (C, τ) one can recover τ from $\mathsf{Sh}(\mathsf{C}, \tau) \subset \mathsf{PSh}(\mathsf{C})$. Applying this fact to the identity $j(X) \to j(X)$ and recalling that $\mathsf{C}_{/X} \subset \mathsf{C}_{/X}$ is a covering sieve (Definition 1.7.2 (1) (a)), we find that the composition

$$\mathsf{C} \xrightarrow{j} \mathsf{PSh}(\mathsf{C}) \xrightarrow{L} \mathsf{Sh}(\mathsf{C}, \tau) \tag{C.2.1}$$

is a fully faithful functor of ∞-categories. Thus, we may say that τ is sub-canonical for the ∞-topos $\mathcal{S}(\mathsf{C}, \tau)$. (using the terminology in the ordinary Grothendieck topology).

Definition C.2.1. The composition (C.2.1) will also be called the *Yoneda embedding*, and will be denoted by the same symbol $j : \mathsf{C} \to \mathsf{Sh}(\mathsf{C}, \tau)$.

C.3. **Hypercomplete ∞-topoi.** The ∞-topoi of τ-sheaves discussed in the previous part has a distinguished property among general ∞-topoi.

Definition C.3.1 ([Lur1, Definition 6.5.1.10, §6.5.2]). Let T be an ∞-topos.

(1) Let $n \in \mathbb{N} \cup \{\infty\}$. A morphism $f : X \to Y$ in an ∞-topos T is called *n-connective* if it is an effective epimorphism (Definition 1.8.10) and $\pi_k(f)$ is trivial for each $k = 0, 1, \ldots, n$.

(2) An object $X \in \mathsf{T}$ is called *hypercomplete* if it is local (Definition C.1.1) with respect to the class of ∞-connective morphisms. We denote by $\mathsf{T}^\wedge$ the sub-∞-category of T spanned by hypercomplete objects.

(3) T is called *hypercomplete* if $\mathsf{T}^\wedge = \mathsf{T}$.

Next we introduce the notion of hypercoverings following [Lur1, §6.5.3]. See also [TVe1, §3.2] for a model-theoretic explanation.

Let us recall the category $\mathbf{\Delta}$ of combinatorial simplices (§1.1) and the nerve $\mathsf{N}(\mathrm{C})$ of an ordinary category C (Definition 1.2.1).

Definition ([Lur1, Definition 6.1.2.2]). A *simplicial object in an ∞-category* C is a simplicial map $U_\bullet : \mathsf{N}(\mathbf{\Delta})^{\mathrm{op}} \to \mathsf{C}$. The ∞-category of simplicial objects in C is denoted by C_Δ.

Following [Lur1, Notation 6.5.3.1], for each $n \in \mathbb{N}$, we denote by $\mathbf{\Delta}^{\le n}$ the full subcategory of $\mathbf{\Delta}$ spanned by $\{[0], \ldots, [n]\}$. If C is a presentable ∞-category, then the restriction functor $\mathrm{sk}_n : \mathsf{C}_\Delta \to \mathsf{Fun}(\mathsf{N}(\mathbf{\Delta}^{\le n})^{\mathrm{op}}, \mathsf{C})$ has a right adjoint:

$$\mathrm{sk}_n : \mathsf{C}_\Delta \rightleftarrows \mathsf{Fun}(\mathsf{N}(\mathbf{\Delta}^{\le n})^{\mathrm{op}}) : r. \tag{C.3.1}$$

In fact, r is given by the right Kan extension [Lur1, §4.3.2] along the inclusion functor $\mathsf{N}(\boldsymbol{\Delta}^{\le n})^{\mathrm{op}} \hookrightarrow \mathsf{N}(\boldsymbol{\Delta})^{\mathrm{op}}$. We set

$$\operatorname{cosk}_n := r \circ \operatorname{sk}_n : \mathsf{C}_{\Delta} \longrightarrow \mathsf{C}_{\Delta}$$

and call it the *n-coskeleton functor*.

Recalling that an ∞-topos is presentable (Fact 1.8.3), we have the following definition.

Definition C.3.2 ([Lur1, Definition 6.5.3.2])**.** Let T be an ∞-topos. A simplicial object $U_\bullet \in \mathsf{T}_\Delta$ is called a *hypercovering* of T if for each $n \in \mathbb{N}$ the unit map $U_n \to (\operatorname{cosk}_{n-1} U_\bullet)_n$ coming from the adjoint (C.3.1) is an effective epimorphism (Definition 1.8.10).

As noted in [Lur1, Remark 6.5.3.3], a hypercovering $U_\bullet$ of T is a simplicial object such that the morphisms $U_0 \to \mathbf{1}_{\mathsf{T}}$, $U_1 \to U_0 \times U_0$, $U_2 \to U_0 \times U_0 \times U_0$, ... are effective epimorphisms. Here $\mathbf{1}_{\mathsf{T}}$ denotes a final object of T.

Next we give the definition of a geometric realization of a simplicial object. We denote by $\boldsymbol{\Delta}_+$ the category of possibly empty finite linearly ordered sets. We can regard $\boldsymbol{\Delta} \subset \boldsymbol{\Delta}_+$ as a full subcategory.

Definition ([Lur1, Notation 6.1.2.12])**.** Let C be an ∞-category and $U_\bullet \in \mathsf{C}_\Delta$. Regarding $U_\bullet$ as a diagram in C indexed by $\mathsf{N}(\boldsymbol{\Delta})^{\mathrm{op}}$, we denote by

$$|U_\bullet| : \mathsf{N}(\boldsymbol{\Delta}_+)^{\mathrm{op}} \longrightarrow \mathsf{C}$$

a colimit for $U_\bullet$ if it exists, and call it a *geometric realization of* $U_\bullet$.

Remark. (1) As noted in [Lur1, Remark 6.1.2.13], $|U_\bullet|$ is determined up to contractible ambiguity.
(2) For a hypercovering $U_\bullet$ of an ∞-topos T, a geometric realization of $U_\bullet$ always exists since T admits arbitrary colimits (Corollary 1.8.4), so that the notation $|U_\bullet|$ makes sense. By the item (1), we call it *the* geometric realization of $U_\bullet$.

Definition. A hypercovering $U_\bullet$ of an ∞-topos T is called *effective* if $|U_\bullet|$ is a final object of T.

Let us cite a criterion for an ∞-topos to be hypercomplete.

Fact C.3.3 ([Lur1, Theorem 6.3.5.12])**.** For an ∞-topos T, the following two conditions are equivalent.
(1) T is hypercomplete.
(2) For each $X \in \mathsf{T}$, every hypercovering $U_\bullet$ of $\mathsf{T}_{/X}$ is effective.

Now we have the following result. See also [TVe1, Theorem 3.4.1] for a discussion in a model-theoretical context.

Fact C.3.4 ([Lur1, Corollary 6.5.3.13])**.** Let T be an ∞-topos. Define S to be the collection of morphisms $|U_\bullet| \to X$ where $U_\bullet$ is a hypercovering of $\mathsf{T}_{/X}$ for an object $X \in \mathsf{T}$. Then an object of T is hypercomplete if and only if it is S-local (Definition C.1.1).

This fact implies

Corollary C.3.5. For an ∞-site (C, τ), the ∞-topos $\mathsf{Sh}(\mathsf{C}, \tau)$ of sheaves is hypercomplete.

Combining with the Yoneda embedding $j : \mathsf{C} \to \mathsf{Sh}(\mathsf{C}, \tau)$ (Definition C.2.1), we also have

Corollary C.3.6. For an ∞-site (C, τ), each object of $\mathsf{Sh}(\mathsf{C}, \tau)$ is equivalent to a colimit of objects in the sub-∞-category $j(\mathsf{C}) \subset \mathsf{Sh}(\mathsf{C}, \tau)$.

Proof. Given a sheaf $F \in \mathsf{Sh}(\mathsf{C}, \tau)$, we can take a hypercovering $U_\bullet$ in $\mathsf{PSh}(\mathsf{C})_{/F}$ such that $|U_\bullet| \to F$ is equivalence by Corollary C.3.5 and Fact C.3.3. Each $U_n \in \mathsf{PSh}(\mathsf{C})$ is equivalent to a colimit of objects in $j(\mathsf{C})$ by Fact 1.5.3. Thus F is equivalent to a colimit of objects in $j(\mathsf{C})$. □

C.4. **Proper base change in an ∞-topos.** In this subsection we recall the proper base change theorem in an ∞-topos [Lur1, §7.3.1].

Recall the ∞-category Cat_∞ of small ∞-category. It admits small limits and small colimits. Now the following definition makes sense.

Definition C.4.1 ([Lur1, Definition 7.3.1.1, 7.3.1.2])**.** A diagram

$$\begin{array}{ccc} \mathsf{B}' & \xrightarrow{q'_*} & \mathsf{C}' \\ {\scriptstyle p'_*}\downarrow & & \downarrow{\scriptstyle p_*} \\ \mathsf{B} & \xrightarrow[q_*]{} & \mathsf{C} \end{array}$$

of ∞-categories is *left adjointable* if the corresponding diagram

$$\begin{array}{ccc} \mathrm{h}\,\mathsf{B}' & \xrightarrow{q'_*} & \mathrm{h}\,\mathsf{C}' \\ {\scriptstyle p_*}\downarrow & & \downarrow{\scriptstyle q_*} \\ \mathrm{h}\,\mathsf{B} & \xrightarrow[q_*]{} & \mathrm{h}\,\mathsf{C} \end{array}$$

of the homotopy categories commutes up to a specified isomorphism $\eta : p_* q'_* \to q_* p'_*$, the functors q_*, q'_* of categories admit left adjoints q^*, ${q'}^*$ and the morphism

$$\alpha : q^* p_* \xrightarrow{u} q^* p_* q'_* {q'}^* \xrightarrow{\eta} q^* q_* p'_* {q'}^* \xrightarrow{c} p'_* {q'}^*$$

is an isomorphism of functors. Here u is the unit and c is the counit associated to each adjunction. We call α the *base change morphism*.

Definition ([Lur1, Definition 7.3.1.4]). A geometric morphism $p : \mathsf{U} \to \mathsf{T}$ of ∞-topoi corresponding to the adjoint pair $p^* : \mathsf{U} \rightleftarrows \mathsf{T} : p_*$ is *proper* if for any cartesian rectangle

$$\begin{array}{ccccc} \mathsf{T}'' & \longrightarrow & \mathsf{T}' & \longrightarrow & \mathsf{T} \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle p_*} \\ \mathsf{U}'' & \longrightarrow & \mathsf{U}' & \longrightarrow & \mathsf{U} \end{array}$$

of ∞-topoi, the left square is left adjointable.

Thus a proper geometric morphism is defined to be one for which the base change theorem holds.
We collect some formal properties of proper geometric morphisms.

Fact ([Lur1, Proposition 7.3.1.6]). (1) Any equivalence of ∞-topoi is a proper geometric morphism.
(2) The class of proper geometric morphisms is closed under equivalence, pullback by any morphism and composition.

In [Lur1, Theorem 7.3.16] it is shown that for a proper map $p : X \to Y$ of topological spaces with X completely regular (i.e., homeomorphic to a subspace of a compact Hausdorff space), the associated geometric morphism $p_* : \mathsf{Sh}(X) \to \mathsf{Sh}(Y)$ is proper.

Appendix D. Stable ∞-categories

D.1. **Definition of stable ∞-categories.** In this subsection we cite from [Lur2, Chap. 1] the necessary notion and statements on stable ∞-category.

Definition ([Lur2, Definition 1.1.1.1]). A *zero object* of an ∞-category C is an object which is both initial and final in the sense of Definition 1.3.7.

Definition D.1.1 ([Lur2, Definition 1.1.1.4, 1.1.1.6]). Let C be an ∞-category with a zero object 0.

(1) A *triangle* in C is a square of the form

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & Z \end{array}$$

We sometimes denote such a triangle simply by $X \to Y \to Z$.

(2) A triangle is called a *fiber sequence* (resp. *cofiber sequence*) if it is a pullback square (resp. pushout square) in the sense of Definition B.4.5.

(3) Let $f : X \to Y$ be a morphism in C. A *fiber* of f is a fiber sequence of the form

$$\begin{array}{ccc} W & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle f} \\ 0 & \longrightarrow & Y \end{array}$$

A *cofiber* of f is a cofiber sequence of the form

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & Z \end{array}$$

The word triangle in the above definition will be used only in this subsection (in fact, up to Definition D.1.2). Note that in the main text we use the word triangle in C to mean a simplicial map $\Delta^2 \to \mathsf{C}$.

Definition D.1.2 ([Lur2, Definition 1.1.1.9])**.** An ∞-category C is called *stable* if it satisfies the following three conditions.

- C has a zero object $0 \in \mathsf{C}$.
- Every morphism in C has a fiber and a cofiber.
- A triangle in C is a fiber sequence if and only if it is a cofiber sequence.

For a stable ∞-category C one can define the suspension functor $\Sigma : \mathsf{C} \to \mathsf{C}$ and the loop functor $\Omega : \mathsf{C} \to \mathsf{C}$ as follows [Lur2, §1.1.2].

Let us assume for a while only that C has a zero object. Consider the full sub-∞-category M^{Σ} of $\mathsf{Fun}(\Delta^1 \times \Delta^1, \mathsf{C})$ spanned by the pushout squares of the form

$$\begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0' & \longrightarrow & Y \end{array}$$

with 0 and $0'$ zero objects of C. As explained in [Lur2, p.23], if morphisms in C have cofibers, then the evaluation at X induces a trivial fibration $i : \mathsf{M}^{\Sigma} \to \mathsf{C}$. Let $s : \mathsf{C} \to \mathsf{M}^{\Sigma}$ be a section of i. Let also $f : \mathsf{M}^{\Sigma} \to \mathsf{C}$ be the functor given by evaluation at Y.

Definition D.1.3. Let C be an ∞-category which has a zero object and where every morphism has a cofiber. The *suspension functor* $\Sigma = \Sigma_{\mathsf{C}} : \mathsf{C} \to \mathsf{C}$ of C is defined to be the composition $\Sigma := f \circ s$ of f and s constructed above.

Dually, denote by M^{Ω} the full sub-∞-category of $\mathsf{Fun}(\Delta^1 \times \Delta^1, \mathsf{C})$ spanned by the pullback squares of the above form. If morphisms in C have fibers, then the evaluation at the vertex Y induces a trivial fibration $f' : \mathsf{M}^{\Sigma} \to \mathsf{C}$. Let $s' : \mathsf{C} \to \mathsf{M}^{\Omega}$ be a section of f'. Let also $i' : \mathsf{M}^{\Omega} \to \mathsf{C}$ be the functor given by the evaluation at X.

Definition. Let C be an ∞-category which has a zero object and where every morphism has a fiber. The *loop functor* $\Omega = \Omega_{\mathsf{C}} : \mathsf{C} \to \mathsf{C}$ of C is defined to be the composition $\Omega := i' \circ s'$ of i' and s' constructed above.

Fact. If C is stable, then $\mathsf{M}^{\Sigma} = \mathsf{M}^{\Omega}$, so that Σ and Ω are mutually inverse equivalences on C.

Definition D.1.4. Let C be a stable ∞-category. For $n \in \mathbb{N}$, we denote by $X \mapsto X[n]$ the n-th power of the suspension functor Σ, and by $X \mapsto X[-n]$ the n-th power of the loop functor Ω. We call them *translations* or *shifts* on C.

We also have the equivalence on the homotopy category $\mathrm{h}\,\mathsf{C}$ induced by $[n] : \mathsf{C} \to \mathsf{C}$, which will be denoted by the same symbol $[n]$ and called translations on $\mathrm{h}\,\mathsf{C}$.

We cite from [Lur2] a construction of new stable ∞-category from old one.

Fact ([Lur2, Propsition 1.1.3.1, Lemma 1.1.3.3])**.** Let C be a stable ∞-category.

(1) For a simplicial set K, the ∞-category $\mathsf{Fun}(K, \mathsf{C})$ of functors is stable.
(2) Let $\mathsf{C}' \subset \mathsf{C}$ be a full sub-∞-category which is stable under cofibers and translations. Then C' is a stable subcategory of C.

We now recall the structure of a triangulated category on the homotopy category $\mathrm{h}\,\mathsf{C}$. A diagram in $\mathrm{h}\,\mathsf{C}$ of the form

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

is called a *distinguished triangle* if there exists a diagram $\Delta^1 \times \Delta^2 \to \mathsf{C}$ of the form

$$\begin{array}{ccccc} X & \xrightarrow{\widetilde{f}} & Y & \longrightarrow & 0 \\ \downarrow & & \downarrow g & & \downarrow \\ 0' & \longrightarrow & Z & \xrightarrow[\widetilde{h}]{} & W \end{array}$$

satisfying the following four conditions:

- $0, 0' \in \mathsf{C}$ are zero objects.
- Both squares are pushout square in C.
- The maps $\widetilde{f}$ and $\widetilde{g}$ in C represent f and g in $\mathrm{h}\,\mathsf{C}$ respectively.
- h is the composition of the homotopy class of $\widetilde{h}$ with the equivalence $W \simeq X[1]$ determined by the outer rectangle.

Fact D.1.5 ([Lur2, Theorem 1.1.2.14])**.** Let C be a stable ∞-category. Then the translations on $\mathrm{h}\,\mathsf{C}$ and the distinguished triangles give $\mathrm{h}\,\mathsf{C}$ the structure of a triangulated category.

By this fact, we know in particular that $\mathrm{h}\,\mathsf{C}$ has the structure of an additive category. For objects X, Y of a stable ∞-category C and an integer n, we denote by $\mathrm{Ext}^n_{\mathsf{C}}(X, Y)$ the abelian group $\mathrm{Hom}_{\mathrm{h}\,\mathsf{C}}(X[-n], Y)$.

Finally we recall the notion of exact functors of stable ∞-categories. See [Lur2, §1.1.4] for the detail.

Definition D.1.6. A functor $F : \mathsf{C} \to \mathsf{C}'$ of stable ∞-categories is called *exact* if the following two conditions are satisfied.

- F carries zero objects in C to zero objects in C'.
- F carries fiber sequences to fiber sequences.

This notion of exactness is compatible with Definition B.10.1 by the following fact.

Fact ([Lur2, Proposition 1.1.4.1])**.** For a functor $F : \mathsf{C} \to \mathsf{C}'$ of stable ∞-categories, the following three conditions are equivalent.

(i) F commutes with finite limits.
(ii) F commutes with finite colimits.
(iii) F is exact in the sense of Definition D.1.6.

D.2. ***t*-structure.** We collect the basics of t-structure on a stable ∞-category citing from [Lur2, §1.2].

Recall the notion of t-structure on a triangulated category in the sense of [BBD]:

Definition D.2.1. A *t-structure* on a triangulated category D is a pair $(\mathrm{D}_{\le 0}, \mathrm{D}_{\ge 0})$ of full subcategories satisfying the following three conditions.

(i) For any $X \in \mathrm{D}_{\ge 0}$ and $Y \in \mathrm{D}_{\le 0}$, we have $\mathrm{Hom}_{\mathrm{D}}(X, Y[-1]) = 0$.
(ii) We have $\mathrm{D}_{\ge 0}[1] \subset \mathrm{D}_{\ge 0}$ and $\mathrm{D}_{\le 0}[-1] \subset \mathrm{D}_{\le 0}$.
(iii) For any $X \in \mathrm{D}$, there exists a fiber sequence $X' \to X \to X'$ in the nerve $\mathsf{N}(\mathrm{D})$ (see Definition D.1.1), where $X' \in \mathrm{D}_{\ge 0}$ and $X'' \in \mathrm{D}_{\le 0}[-1]$.

Definition D.2.2 ([Lur2, Definition 1.2.1.4])**.** (1) Let C be an ∞-category. A t-structure on C is a t-structure on the homotopy category $\mathrm{h}\,\mathsf{C}$.
(2) Let $n \in \mathbb{Z}$. Given a t-structure on C, we denote by $\mathsf{C}_{\ge n}$ and $\mathsf{C}_{\le n}$ the full sub-∞-categories spanned by those objects belonging to $(\mathrm{h}\,\mathsf{C})_{\ge n}$ and $(\mathrm{h}\,\mathsf{C})_{\le n}$, respectively.

By [Lur2, Proposition 1.2.1.5], for a stable ∞-category C equipped with a t-structure, the full sub-∞-categories $\mathsf{C}_{\ge n}$ is a localization of C. Thus, following [Lur2, Notation 1.2.1.7], it makes sense to define the functor

$$\tau_{\le n} : \mathsf{C} \longrightarrow \mathsf{C}_{\le n}$$

to be a left adjoint to the inclusion $\mathsf{C}_{\le n} \hookrightarrow \mathsf{C}$. We also denote by $\tau_{\ge n} : \mathsf{C} \to \mathsf{C}_{\ge n}$ a right adjoint to the inclusion $\mathsf{C}_{\ge n} \hookrightarrow \mathsf{C}$.

Remark ([Lur2, Remark 1.2.1.3])**.** A t-structure on a stable ∞-category C is determined by either of the corresponding localizations $\mathsf{C}_{\le 0}, \mathsf{C}_{\ge 0} \subset \mathsf{C}$. We call it the *t-structure determined by* $(\mathsf{C}_{\le 0}, \mathsf{C}_{\ge 0})$.

As in the case of non-∞-categorical case, we have the bounded sub-∞-categories.

Definition D.2.3. Let C be an ∞-category equipped with a t-structure. We define the sub-∞-categories $\mathsf{C}^+, \mathsf{C}^-, \mathsf{C}^b$ of C by

$$\mathsf{C}^+ := \cup_n \mathsf{C}_{\le n}, \quad \mathsf{C}^- := \cup_n \mathsf{C}_{\ge -n}, \quad \mathsf{C}^b := \mathsf{C}^+ \cap \mathsf{C}^-.$$

We call C to be *left bounded* if $\mathsf{C} = \mathsf{C}^+$, *right bounded* if $\mathsf{C} = \mathsf{C}^-$, and *bounded* if $\mathsf{C} = \mathsf{C}^b$,

Note that these sub-∞-categories C^* $(* \in \{\pm, b\})$ are stable.

By [Lur2, Proposition 1.2.1.10] we have an equivalence $\tau_{\le m} \circ \tau_{\ge n} \to \tau_{\ge n} \circ \tau_{\le m}$ of functors $\mathsf{C} \to \mathsf{C}_{\le m} \cap \mathsf{C}_{\ge n}$ as in the non-derived case [BBD, §1].

Definition D.2.4 ([Lur2, Notation 1.2.1.7])**.** (1) We define the *heart* of C to be

$$\mathsf{C}^{\heartsuit} := \mathsf{C}_{\le 0} \cap \mathsf{C}_{\ge 0} \subset \mathsf{C}.$$

(2) We define $\pi_0 := \tau_{\le 0} \circ \tau_{\ge 0} : \mathsf{C} \to \mathsf{C}^{\heartsuit}$
(3) For $n \in \mathbb{Z}$ we define $\pi_n : \mathsf{C} \to \mathsf{C}^{\heartsuit}$ to be the composition of π_0 with the shift functor $X \mapsto X[-n]$.

For later use, we introduce

Definition ([Lur2, §1.2.1, p.44])**.** Let C be a stable ∞-category equipped with a t-structure. The *left completion* $\widehat{\mathsf{C}}$ of C is a limit of the tower of ∞-categories

$$\cdots \xrightarrow{\tau_{\le 2}} \mathsf{C}_{\le 2} \xrightarrow{\tau_{\le 1}} \mathsf{C}_{\le 1} \xrightarrow{\tau_{\le 0}} \mathsf{C}_{\le 0} \xrightarrow{\tau_{\le -1}} \cdots .$$

As explained in loc. cit., we have the following description of $\widehat{\mathsf{C}}$. Let $\mathsf{N}(\mathbb{Z})$ denote the nerve of (the category associated to) the linearly ordered set $\mathbb{Z}$. Then $\widehat{\mathsf{C}}$ is the full sub-∞-category of $\mathsf{Fun}(\mathsf{N}(\mathbb{Z}), \mathsf{C})$ spanned by $F : \mathsf{N}(\mathbb{Z}) \to \mathsf{C}$ such that $F(n) \in \mathsf{C}_{\le -n}$ for each $n \in \mathbb{Z}$ and the morphism $\tau_{\le -n} F(m) \to F(n)$ induced by $F(m) \to F(n)$ is an equivalence for each pair $m \le n$.

Fact ([Lur2, Proposition 1.2.1.17])**.** Let C be a stable ∞-category equipped with a t-structure.
(1) The left completion $\widehat{\mathsf{C}}$ is stable.
(2) $\widehat{\mathsf{C}}$ has a t-structure determined by $(\widehat{\mathsf{C}}_{\le 0}, \widehat{\mathsf{C}}_{\ge 0})$, where $\widehat{\mathsf{C}}_{\le 0}$ and $\widehat{\mathsf{C}}_{\ge 0}$ are full sub-∞-categories of $\widehat{\mathsf{C}}$ spanned by functors factoring through $\mathsf{C}_{\le 0}$ and $\mathsf{C}_{\ge 0}$ respectively.
(3) There is a canonical functor $\mathsf{C} \to \widehat{\mathsf{C}}$, which is exact and induces an equivalence $\mathsf{C}_{\ge 0} \to \widehat{\mathsf{C}}_{\ge 0}$.

Definition D.2.5. A stable ∞-category C equipped with a t-structure is called *left complete* if the canonical functor $\mathsf{C} \to \widehat{\mathsf{C}}$ is an equivalence. The *right completion* and the *right completeness* are defined dually.

We also give some properties of a t-structure.

Definition D.2.6. Let C be a stable ∞-category equipped with a t-structure determined by $(\mathsf{C}_{\le 0}, \mathsf{C}_{\ge 0})$.
(1) The t-structure is *accessible* if $\mathsf{C}_{\ge 0}$ is a presentable ∞-category.
(2) The t-structure is *compatible with filtered colimits* if the ∞-category $\mathsf{C}_{\le 0}$ is stable under filtered colimits.
(3) Assume that C is a symmetric monoidal ∞-category. The t-structure is *compatible with the symmetric monoidal structure* if $\mathsf{C}_{\ge 0}$ contains the unit object of C and is stable under tensor product.

D.3. **Derived ∞-category.** Following [Lur2, §1.3] we recall a construction of stable ∞-category from an abelian category. We will use the cohomological notation of complexes, opposed to the homological notation in loc. cit. Thus a complex $M = (M^*, d)$ is a sequence of morphisms

$$\cdots \xrightarrow{d^{-2}} M^{-1} \xrightarrow{d^{-1}} M^0 \xrightarrow{d^0} M^1 \xrightarrow{d^1} \cdots$$

such that $d^{n+1} \circ d^n = 0$ for any $n \in \mathbb{Z}$.

D.3.1. *Construction.* Let D be a dg-category over a commutative ring k (Definition 9.1.1). For $X, Y \in \mathrm{D}$, we denote the complex of morphisms from X to Y by $\mathrm{Hom}_{\mathrm{D}}(X, Y) = (\mathrm{Hom}_{\mathrm{D}}(X, Y)^*, d)$.

Definition D.3.1 ([Lur2, Construction 1.3.16])**.** For a dg-category D, we define a simplicial set $\mathsf{N}_{\mathrm{dg}}(\mathrm{D})$ as follows. For $n \in \mathbb{N}$, we define $\mathsf{N}_{\mathrm{dg}}(\mathrm{D})_n$ to be the set consisting of $(\{X_i \mid 0 \le i \le n\}, \{f_I\})$ where
- X_i is an object of D.
- For each subset $I = \{i_- < i_m < \cdots < i_1 < i_+\} \subset [n]$ with $m \in \mathbb{N}$, f_I is an element of the k-module $\mathrm{Hom}_{\mathrm{D}}(X_{i_-}, X_{i_+})^{-m}$ satisfying $df_I = \sum_{1 \le j \le m} (-1)^j (f_{I \setminus \{i_j\}} - f_{\{i_j < \cdots < i_1 < i_+\}} \circ f_{\{i_- < i_m < \cdots < i_j\}})$.

We omit the description of face and degeneracy maps. $\mathsf{N}_{\mathrm{dg}}(\mathrm{D})$ is called the *differential graded nerve*, or the *dg-nerve*, of D.

As explained in [Lur2, Example 1.3.1.8], lower simplices of $\mathsf{N}_{\mathrm{dg}}(\mathrm{D})$ are given by

- A 0-simplex of $\mathsf{N}_{\mathrm{dg}}(\mathrm{D})$ is an object of D.
- A 1-simplex is a morphism $f : X_0 \to X_1$ of D, i.e., an element $f \in \mathrm{Hom}_{\mathrm{D}}(X_0, X_1)^0$ with $df = 0$.
- A 2-simplex consists of $X_0, X_1, X_2 \in \mathrm{D}$, $f_{ij} \in \mathrm{Hom}_{\mathrm{D}}(X_i, X_j)^0$ for $(i,j) = (0,1), (1,2), (0,2)$ with $df_{ij} = 0$, and $g \in \mathrm{Hom}_{\mathrm{D}}(X_0, X_2)^{-1}$ with $dg = f_{01} \circ f_{12} - f_{02}$.

Fact ([Lur2, Proposition 1.3.1.10])**.** For any dg-category D, the simplicial set $\mathsf{N}_{\mathrm{dg}}(\mathrm{D})$ is an ∞-category.

Notation. Let A be an additive category.

(1) C(A) is the category of complexes in A. It has a natural structure of a dg-category over $\mathbb{Z}$, and hereafter we consider C(A) as a dg-category.
(2) $\mathrm{C}^+(\mathrm{A}) \subset \mathrm{C}(\mathrm{A})$ is the full subcategory spanned by complexes bounded below, i.e., spanned by those M such that $M^n = 0$ for $n \ll 0$.
(3) $\mathrm{C}^-(\mathrm{A}) \subset \mathrm{C}(\mathrm{A})$ is the full subcategory spanned by complexes M bounded above, i.e., $M^n = 0$ for $n \gg 0$.

For the dg-category C(A) we can construct the dg-nerve $\mathsf{N}_{\mathrm{dg}}(\mathrm{C}(\mathrm{A}))$. We can now introduce the derived ∞-category for abelian category with enough projective or injective objects.

Definition D.3.2 ([Lur2, Definition 1.3.2.7, Variant 1.3.2.8])**.** Let A be an abelian category.

(1) Assume A has enough injective objects, and let $\mathrm{A}_{\mathrm{inj}} \subset \mathrm{A}$ be the full subcategory spanned by injective objects. We define the ∞-category $\mathsf{D}^+_\infty(\mathrm{A})$ to be
$$\mathsf{D}^+_\infty(\mathrm{A}) := \mathsf{N}_{\mathrm{dg}}(\mathrm{C}^+(\mathrm{A}_{\mathrm{inj}}))$$
and call it the *lower bounded derived ∞-category* of A.
(2) Assume A has enough projective objects, and let $\mathrm{A}_{\mathrm{proj}} \subset \mathrm{A}$ be the full subcategory spanned by projective objects. We define the ∞-category $\mathsf{D}^-_\infty(\mathrm{A})$ to be
$$\mathsf{D}^-_\infty(\mathrm{A}) := \mathsf{N}_{\mathrm{dg}}(\mathrm{C}^-(\mathrm{A}_{\mathrm{proj}}))$$
and call it the *upper bounded derived ∞-category* of A.

As noted in [Lur2, Variant 1.3.2.8], we have an equivalence $\mathsf{D}^+_\infty(\mathrm{A})^{\mathrm{op}} \simeq \mathsf{D}^-_\infty(\mathrm{A}^{\mathrm{op}})$. We will mainly discuss on $\mathsf{D}^+_\infty(\mathrm{A})$ with A enough injectives hereafter.

For any additive category A, the ∞-category $\mathsf{N}_{\mathrm{dg}}(\mathrm{C}(\mathrm{A}))$ is stable by [Lur2, Proposition 1.3.2.10]. Then one can deduce

Fact ([Lur2, Corollary 1.3.2.18])**.** For an abelian category A with enough injective objects, the ∞-category $\mathsf{D}^+_\infty(\mathrm{A})$ is stable.

We also have a description of $\mathsf{D}^\pm_\infty(\mathrm{A})$ by localization (Definition B.7.1), which is similar to the definition of the ordinary derived category.

Fact ([Lur2, Theorem 1.3.4.4, Proposition 1.3.4.5])**.** Let A be an abelian category with enough injective objects. We denote by W be the collection of those morphisms in $\mathrm{C}^+(\mathrm{A})$ (regarded as an ordinary category) which are quasi-isomorphisms of complexes, and by W_{dg} quasi-isomorphisms in $\mathsf{N}_{\mathrm{dg}}(\mathrm{C}^+(\mathrm{A}))$. Then we have canonical equivalences
$$\mathsf{N}(\mathrm{C}^+(\mathrm{A}))[W^{-1}] \simeq \mathsf{N}_{\mathrm{dg}}(\mathrm{C}^+(\mathrm{A}))[W_{\mathrm{dg}}^{-1}] \simeq \mathsf{D}^+_\infty(\mathrm{A}).$$
A dual description exists for an abelian category with enough projective objects.

D.3.2. *Grothendieck abelian category.* Following the terminology of [Lur1, Lur2], we say a category C is *presentable* if it satisfies the following two conditions.

- C admits arbitrary small limits and small colimits.
- C is generated under small colimits by a set of κ-compact objects (Definition B.8.2) for some regular cardinal number κ.

See Fact B.8.4 for the related claim on presentable ∞-categories.

Now let us recall

Definition D.3.3 ([Lur2, Definition 1.3.5.1])**.** An abelian category A is called *Grothendieck* if it satisfies the following conditions.

- A is presentable as an ordinary category.
- The collection of monomorphisms in A is closed under small filtered colimits.

Let us also recall a model structure of the dg-category C(A) of complexes in A.

Fact D.3.4 ([Lur2, Propsoition 1.3.5.3]). Let A be a Grothendieck abelian category. Then the category C(A) has the following model structure.

- A morphism $f : M \to N$ in C(A) is a cofibration if for any $k \in \mathbb{Z}$ the map $f : M_k \to N_k$ is a monomorphism in A.
- A morphism $f : M \to N$ in C(A) is a weak equivalence if it is a quasi-isomorphism.

We call it the *injective model structure* of C(A).

We have the following characterization of fibrant objects in this model category.

Fact. Let A be a Grothendieck abelian category and $M \in \mathrm{C(A)}$. If M is fibrant, then each M_n is an injective object of A. Conversely, if each M_n is injective and $M_n \simeq 0$ for $n \gg 0$, then M is fibrant.

As a corollary, one can reprove that a Grothendieck abelian category has enough injective objects [Lur2, Corollary 1.3.5.7].

As in Definition B.6.2, we denote by A° the subcategory of fibrant-cofibrant objects in C(A) with the injective model structure.

Definition D.3.5 ([Lur2, Definition 1.3.5.8]). Let A be a Grothendieck abelian category. We define an ∞-category $\mathsf{D}_\infty(\mathrm{A})$ to be

$$\mathsf{D}_\infty(\mathrm{A}) := \mathsf{N}_{\mathrm{dg}}(\mathrm{C}(\mathrm{A})^\circ)$$

and call it the *unbounded derived* ∞*-category* of A.

Here is a list of properties of the unbounded derived ∞-category $\mathsf{D}_\infty(\mathrm{A})$.

Fact D.3.6. Let A be a Grothendieck abelian category.

(1) $\mathsf{D}_\infty(\mathrm{A})$ is stable [Lur2, Proposition 1.3.5.9].
(2) The natural inclusion $\mathsf{D}_\infty(\mathrm{A}) \hookrightarrow \mathsf{N}_{\mathrm{dg}}(\mathrm{C}(\mathrm{A}))$ has a left adjoint L which is a localization functor (Definition B.7.2) [Lur2, Proposition 1.3.5.13].
(3) $\mathsf{D}_\infty(\mathrm{A})$ is equivalent to the underlying ∞-category $\mathsf{N}_{\mathrm{spl}}(\mathrm{C}(\mathrm{A})^\circ)$ of C(A) regarded as a discrete simplicial model category [Lur2, Proposition 1.3.5.13].
(4) $\mathsf{D}_\infty(\mathrm{A})$ is presentable as an ∞-category [Lur2, Proposition 1.3.5.21 (1)].

D.3.3. *t-structure.* Let us cite from [Lur2, §1.3] the natural t-structures on the derived ∞-categories.

Fact ([Lur2, Proposition 1.3.2.19]). Let $\mathcal{A}$ be an abelian category with enough injective objects. We define $\mathsf{D}^+_\infty(\mathrm{A})_{\geq 0} \subset \mathsf{D}^+_\infty(\mathrm{A})$ to be the full sub-∞-category spanned by those objects $\mathcal{M}$ such that the homology $H_n(\mathcal{M}) \in \mathrm{A}$ vanishes for $n < 0$. We define $\mathsf{D}^+_\infty(\mathrm{A})_{\leq 0}$ similarly. Then the pair

$$\big(\mathsf{D}^+_\infty(\mathrm{A})_{\leq 0}, \mathsf{D}^+_\infty(\mathrm{A})_{\geq 0}\big)$$

determines a t-structure on $\mathsf{D}^+_\infty(\mathrm{A})$, and there is a canonical equivalence $\mathsf{D}^+_\infty(\mathrm{A})^\heartsuit \simeq \mathsf{N}(\mathrm{A})$.

Fact D.3.7 ([Lur2, Proposition 1.3.5.21 (2), (3)]). Let A be a Grothendieck abelian category. We denote by $\mathsf{D}_\infty(\mathrm{A})_{\geq 0} \subset \mathsf{D}_\infty(\mathrm{A})$ the full sub-∞-category spanned by those objects M such that $H_n(M) \simeq 0$ for $n < 0$. $\mathsf{D}_\infty(\mathrm{A})_{\leq 0}$ is defined similarly. Then the pair

$$(\mathsf{D}_\infty(\mathrm{A})_{\leq 0}, \mathsf{D}_\infty(\mathrm{A})_{\geq 0})$$

determines a t-structure on $\mathsf{D}_\infty(\mathrm{A})$, which is accessible, right complete and compatible with filtered colimits (Definition D.2.5, D.2.6).

Let us also cite the following useful result.

Fact D.3.8 ([Lur2, Theorem 1.3.3.2]). Let A be an abelian category with enough injective (resp. projective) objects, C be a stable ∞-category equipped with a left complete t-structure, and $\mathsf{E} \subset \mathsf{Fun}(\mathsf{D}^+_\infty(\mathrm{A}), \mathsf{C})$ (resp. $\mathsf{E} \subset \mathsf{Fun}(\mathsf{D}^-_\infty(\mathrm{A}), \mathsf{C})$) be the full sub-$\infty$-category spanned by those left (resp. right) t-exact functors which carry injective (resp. projective) objects of A into $\mathsf{C}^\heartsuit$. Then the construction

$$F \longmapsto \tau_{\leq 0} \circ \big(F|_{\mathsf{D}^+_\infty(\mathrm{A})^\heartsuit}\big)$$

(resp. $F \mapsto \tau_{\leq 0} \circ (F|_{\mathsf{D}^-_\infty(\mathrm{A})^\heartsuit})$) determines an equivalence from E to the nerve of the category of left (resp. right) exact functors $\mathrm{A} \to \mathsf{C}^\heartsuit$.

Corollary. Let A be an abelian category with enough injective (resp. projective) objects, and C be a stable ∞-category equipped with a right (resp. left) complete t-structure (Definition D.2.5). Then any left (resp. right) exact functor $\mathrm{A} \to \mathsf{C}^\heartsuit$ of abelian categories can be extended to a t-exact functor $\mathsf{D}^+_\infty(\mathrm{A}) \to \mathsf{C}$ (resp. $\mathsf{D}^-_\infty(\mathrm{A}) \to \mathsf{C}$) uniquely up to a canonical equivalence.

Definition D.3.9 ([Lur2, Definition 1.3.3.1]). Let $f : \mathsf{C} \to \mathsf{C}'$ be a functor of stable ∞-categories equipped with t-structures.

(1) f is *left t-exact* if it is exact (Definition B.10.1) and carries $\mathsf{C}_{\le 0}$ into $\mathsf{C}'_{\le 0}$.
(2) f is *right t-exact* if it is exact and carries $\mathsf{C}_{\ge 0}$ into $\mathsf{C}'_{\ge 0}$,
(3) f is *t-exact* if it is both left and right t-exact.

Then we have another corollary of Fact D.3.8 which is standard in the ordinary derived category.

Corollary ([Lur2, Example 1.3.3.4]). Let $f : \mathrm{A} \to \mathrm{B}$ be a functor between abelian categories.

(1) If A and B have enough injective objects and f is right exact, then f extends to a left t-exact functor
$$\mathrm{R}f : \mathsf{D}^+_\infty(\mathrm{A}) \longrightarrow \mathsf{D}^+_\infty(\mathrm{B})$$
which is unique up to contractible ambiguity. We call it the *right derived functor* of f.
(2) Dually, if A and B have enough projective objects and f is right exact, then f extends to a right t-exact functor
$$\mathrm{L}f : \mathsf{D}^-_\infty(\mathrm{A}) \longrightarrow \mathsf{D}^-_\infty(\mathrm{B})$$
which is unique up to contractible ambiguity. We call it the *left derived functor* of f.

Appendix E. Spectra and stable modules

Here we explain spectra, ring spectra and stable modules following [Lur2]. Let us remark that the contents in §E.2 will not be used in the main text except Fact E.2.3.

E.1. **Spectra.** Let us give an important example of a stable ∞-category, the *∞-category* Sp *of spectra.* We need some preliminary definitions.

Definition ([Lur1, Definition 7.2.2.1]). A *pointed object* of an ∞-category C is a morphism $X_* : 1 \to \mathsf{C}$ where 1 is a final object of C (Definition 1.3.7). We denote by C_* the full sub-∞-category of $\mathsf{Fun}(\Delta^1, \mathsf{C})$ spanned by pointed objects of C.

Recall that $\mathcal{S}$ denotes the ∞-category of spaces (Definition 1.4.3). Thus $\mathcal{S}_*$ denotes the ∞-category of pointed objects in $\mathcal{S}$. Noting that $\mathcal{S}$ has a final object, we choose and denote it by $* \in \mathcal{S}$.

Definition ([Lur2, Definition 1.4.2.5]). We denote by $\mathcal{S}^{\mathrm{fin}} \subset \mathcal{S}$ the smallest full sub-∞-category which contains a final object $* \in \mathcal{S}$ and is stable under taking finite colimits. We also denote by $\mathcal{S}^{\mathrm{fin}}_* := (\mathcal{S}^{\mathrm{fin}})_*$ the ∞-category of pointed objects of $\mathcal{S}^{\mathrm{fin}}$.

We need another definition.

Definition ([Lur2, Definition 1.4.2.1]). Let $F : \mathsf{C} \to \mathsf{B}$ be a functor of ∞-categories

(1) Assume C admits pushouts. F is called *excisive* if F carries pushout squares in C to pullback squares in B.
(2) Assume C has a final object $*$. F is called *reduced* if $F(*)$ is a final object of B.

Now we can introduce

Definition E.1.1 ([Lur2, Definition 1.4.2.8, Definition 1.4.3.1]). (1) Let C be an ∞-category admitting finite limits. A *spectrum object* of C is defined to be a reduced excisive functor $F : \mathcal{S}^{\mathrm{fin}}_* \to \mathsf{C}$. We denote by $\mathsf{Sp}(\mathsf{C})$ the full sub-∞-category of $\mathsf{Fun}(\mathcal{S}^{\mathrm{fin}}_*, \mathsf{C})$ spanned by spectrum objects, and call it the *∞-category of spectra in* C.
(2) A *spectrum* is a spectrum object of the ∞-category $\mathcal{S}$ of spaces. We denote by $\mathsf{Sp} := \mathsf{Sp}(\mathcal{S}_*)$ the *∞-category of spectra.*

Fact E.1.2 ([Lur2, Corollary 1.4.2.17]). If C is an ∞-category admitting finite limits, then the ∞-category $\mathsf{Sp}(\mathsf{C})$ is stable.

In particular we have the shift functor $[n]$ with $n \in \mathbb{Z}$ on $\mathsf{Sp}(\mathsf{C})$ (Definition D.1.4). Then we can introduce

Definition E.1.3 ([Lur2, Notation 1.4.2.20]). Let C be an ∞-category admitting finite limits.

(1) Let S^0 be the 0-sphere regarded as an object of $\mathcal{S}^{\mathrm{fin}}_*$. We denote the evaluation functor at $S^0 \in \mathcal{S}^{\mathrm{fin}}_*$ by
$$\Omega^\infty : \mathsf{Sp}(\mathsf{C}) \longrightarrow \mathsf{C}.$$

(2) For $n \in \mathbb{Z}$ we denote by
$$\Omega^{\infty-n} : \mathsf{Sp}(\mathsf{C}) \longrightarrow \mathsf{C}$$
the composition $\Omega^{\infty} \circ [n]$, where $[n]$ denotes the shift functor on $\mathsf{Sp}(\mathsf{C})$.

For $n \in \mathbb{N}$, the functor $\Omega^{\infty-n}$ can be regarded as the evaluation functor at the pointed n-sphere $S^n \in \mathcal{S}_*^{\mathrm{fin}}$. The following fact means that a spectrum object can be regarded as a series of pointed objects together with loop functors as the classical homotopy theory claims.

Fact E.1.4 ([Lur2, Proposition 1.4.2.24, Remark 1.4.2.25])**.** For an ∞-category C admitting finite limits, $\mathsf{Sp}(\mathsf{C})$ is equivalent to the limit of the tower $\cdots \to \mathsf{C}_* \xrightarrow{\Omega} \mathsf{C}_* \xrightarrow{\Omega} \mathsf{C}_*$ of ∞-categories.

Using Ω^{∞} we can endow a t-structure on $\mathsf{Sp}(\mathsf{C})$.

Fact E.1.5 ([Lur2, Proposition 1.4.3.4])**.** Let C be a presentable ∞-category, and $\mathsf{Sp}(\mathsf{C})_{\leq -1} \subset \mathsf{Sp}(\mathsf{C})$ be the full sub-∞-category spanned by those objects X such that $\Omega^{\infty}(X)$ is a final object of C. Then $\mathsf{Sp}(\mathsf{C})_{\leq -1}$ determines a t-structure on $\mathsf{Sp}(\mathsf{C})$.

Let us also recall the sphere spectrum. By [Lur2, Proposition 1.4.4.4], if C is a presentable ∞-category, then the functor $\Omega^{\infty} : \mathsf{Sp}(\mathsf{C}) \to \mathsf{C}$ admits a left adjoint
$$\Sigma^{\infty} : \mathsf{C} \longrightarrow \mathsf{Sp}(\mathsf{C}).$$

Definition E.1.6. For $n \in \mathbb{N}$, we denote the composition with the shift functor $[n]$ on $\mathsf{Sp}(\mathsf{C})$ by
$$\Sigma^{\infty+n} := [n] \circ \Sigma : \mathsf{C} \longrightarrow \mathsf{Sp}(\mathsf{C}).$$
We also denote the image of a final object $1_{\mathsf{C}} \in \mathsf{C}$ under the functor $\Sigma^{\infty+n}$ by
$$S^n_{\mathsf{C}} := \Sigma^{\infty+n}(1_{\mathsf{C}}) \in \mathsf{Sp}(\mathsf{C}).$$

Obviously $\Sigma^{\infty+n}$ is a left adjoint of $\Omega^{\infty-n}$.
In the case $\mathsf{C} = \mathcal{S}$ we have

Fact E.1.7. The stable ∞-category Sp of spectra has the following properties.

(1) Sp is freely generated by the sphere spectrum $S^0_{\mathcal{S}}$ under (small) colimits [Lur2, Corollary 1.4.4.6].
(2) Sp is equipped with the symmetric monoidal structure induced by smash products [Lur2, §4.8.2].
(3) The heart $\mathsf{Sp}^{\heartsuit}$ of the t-structure (Fact E.1.5) is equivalent to the nerve of the ordinary category of abelian groups [Lur2, Proposition 1.4.3.6].

E.2. **Ring spectra.** For $n \in \mathbb{N}$, we denote by $\mathbb{E}_n^{\otimes}$ the ∞-*operad of little n-cubes* in the sense of [Lur2, Definition 5.1.0.2]. We have a natural sequence
$$\mathbb{E}_0^{\otimes} \hookrightarrow \mathbb{E}_1^{\otimes} \hookrightarrow \mathbb{E}_2^{\otimes} \hookrightarrow \cdots$$
of ∞-operads. By [Lur2, Corollary 5.1.1.5], the colimit of this sequence is equivalent to the commutative ∞-operad [Lur2, Example 2.1.1.18], which we denote by $\mathbb{E}_{\infty}^{\otimes}$.

Let k be an $\mathbb{E}_{\infty}$-ring. We denote by $\mathsf{Mod}_k(\mathsf{Sp})$ the ∞-category of k-module spectra (see [Lur2, Notation 7.1.1.1]), which has a symmetric monoidal structure.

Definition E.2.1 ([Lur7, Notation 0.3])**.** Let k be an $\mathbb{E}_{\infty}$-ring. We denote by
$$\mathsf{CAlg}_k = \mathsf{CAlg}_k(\mathsf{Mod}_k(\mathsf{Sp}))$$
the ∞-category of commutative ring objects in the symmetric monoidal ∞-category $\mathsf{Mod}_k(\mathsf{Sp})$, and call an object of CAlg_k a *commutative k-algebra spectrum.* If k is connective, then we denote by $\mathsf{CAlg}_k^{\mathrm{cn}} \subset \mathsf{CAlg}_k$ the full sub-∞-category spanned by connective commutative k-algebra spectra.

An ordinary commutative ring k can be regarded as an $\mathbb{E}_{\infty}$-ring. Let us explain a relationship between the ∞-category $\mathsf{CAlg}_k^{\mathrm{cn}}$ and the ∞-category arising from *simplicial commutative k-algebras.*

Let us denote by sCom_k the category of simplicial commutative k-algebras. In other words, an object of sCom_k is a simplicial object in the category of commutative k-algebras (see §1.1). We have

Fact E.2.2. The category sCom_k has a simplicial model structure (Definition B.6.1) determined by the following data.

- A morphism $A_\bullet \to B_\bullet$ in sCom_k is a weak equivalence if and only if the underlying simplicial map is a weak homotopy equivalence.
- A morphism in sCom_k is a fibration if and only if the underlying simplicial map is a Kan fibration.

We denote by $\mathrm{sCom}_k^\circ \subset \mathrm{sCom}_k$ the full subcategory spanned by fibrant-cofibrant objects. By Definition B.6.2, we have the underlying ∞-category $\mathsf{N}_{\mathrm{spl}}(\mathrm{sCom}_k^\circ)$.

Fact E.2.3 ([Lur2, Proposition 7.1.4.20, Warning 7.1.4.21], [Lur5, Proposition 4.1.11])**.** There is a functor

$$\mathsf{N}_{\mathrm{spl}}(\mathrm{sCom}_k^\circ) \longrightarrow \mathsf{CAlg}_k^{\mathrm{cn}}$$

of ∞-categories which preserves small limits and colimits and which admits left and right adjoints. If moreover the base ring k contains $\mathbb{Q}$, then this functor is an equivalence.

E.3. **Stable modules.** Following [TVe2, §1.2.11, §2.2.1], we introduce some terminology on stable modules of derived rings. We will use some terminology and facts on derived ∞-categories (see §D.3).

Let k be a commutative ring. For a derived k-algebra A, we denote by sMod_A the ∞-category of A-modules in the ∞-category sMod_k of simplicial k-modules (§2.2.1). The ∞-category sMod_A admits finite limits. Thus we have the ∞-category $\mathsf{Sp}(\mathsf{sMod}_A)$ of spectrum objects in sMod_A (Definition E.1.1). It is a stable ∞-category in the sense of Definition D.1.2. We also have the suspension functor $\Sigma^\infty : \mathsf{sMod}_A \to \mathsf{Sp}(\mathsf{sMod}_A)$ in the sense of Definition E.1.6.

Definition E.3.1. For a derived k-algebra A, we denote

$$\mathsf{Sp}(A) := \mathsf{Sp}(\mathsf{sMod}_A)$$

and call it the *∞-category of stable A-modules*. For $n \in \mathbb{Z}$, the n-th shift functor (Definition D.1.4) on $\mathsf{Sp}(A)$ is denoted by $[n]$. We also denote the suspension functor $\Sigma^\infty : \mathsf{sMod}_A \to \mathsf{Sp}(A)$ by

$$\Sigma_A^\infty : \mathsf{sMod}_A \longrightarrow \mathsf{Sp}(A).$$

Let us explain another description of $\mathsf{Sp}(A)$. We start with the recollection of the *normalized chain complex* (see [GJ, Chap. III, §2] for the detail). For a derived k-algebra $A \in \mathrm{sCom}_k$, we define the chain complex $N(A)$ as follows: The graded component is defined to be

$$N(A)_n := \cap_{i=0}^{n-1} \operatorname{Ker} d_i,$$

where $d_i : A_n \to A_{n-1}$ denote the face maps (§1.1). The map defined by

$$(-1)^n d_n : N(A)_n \longrightarrow N(A)_{n-1},$$

gives the differential on $N(A)$ due to the simplicial identity $d_{n-1}d_n = d_n d_{n-1}$. The commutative ring structure on A induces a structure of a commutative k-dg-algebra on the complex $N(A)$.

Definition. The obtained commutative k-dg-algebra $N(A)$ is called the *normalized chain complex* of A.

We denote by $\mathrm{C}(N(A))$ the dg-category of $N(A)$-dg-modules. Considering it as the model category with the injective model structure (Fact D.3.4), we have the subcategory $\mathrm{C}(N(A))^\circ$ of fibrant-cofibrant objects. Then by taking the dg-nerve (Definition D.3.1), we have an ∞-category $\mathsf{N}_{\mathrm{dg}}(\mathrm{C}(N(A))^\circ)$. We will mainly use the next description of $\mathsf{Sp}(A)$ in the following presentation.

Lemma. We have an equivalence of ∞-categories

$$\mathsf{N}_{\mathrm{dg}}(\mathrm{C}(N(A))^\circ) \simeq \mathsf{Sp}(A).$$

Let Λ be a commutative k-algebra. Then we can identify $\mathrm{C}(N(\Lambda)) = \mathrm{C}(\Lambda)$, so that we have

Fact E.3.2. There is an equivalence

$$\mathsf{Sp}(\Lambda) \simeq \mathsf{D}_\infty(\Lambda) := \mathsf{N}_{\mathrm{dg}}(\mathrm{C}(\Lambda)^\circ).$$

We call $\mathsf{D}_\infty(\Lambda)$ the *derived ∞-category of Λ-modules*. See Definition D.3.5 for the detail, where the derived ∞-category is denoted by $\mathsf{D}_\infty(\mathrm{Mod}_\Lambda)$. Taking the homotopy groups in the above equivalence, we have

Corollary.

$$\mathrm{h}\,\mathsf{Sp}(\Lambda) \simeq D(\Lambda) \simeq \mathrm{Ho}\,\mathrm{C}(\Lambda),$$

where $D(\Lambda)$ denotes the derived category of unbounded complexes of Λ-modules (in the ordinary sense) and $\mathrm{Ho}\,\mathrm{C}(\Lambda)$ denotes the homotopy category of the dg-category $\mathrm{C}(\Lambda)$ equipped with the model structure in Fact D.3.4.

We have the theory of t-structures for stable ∞-categories. See §D.2 for an account. Applying Fact D.3.7 to the present situation, we obtain the following t-structure of $\mathsf{Sp}(\Lambda)$.

Lemma E.3.3. Let Λ be a commutative k-algebra. Then $\mathsf{Sp}(\Lambda) \simeq \mathsf{D}_\infty(\Lambda) = \mathsf{N}_{\mathrm{dg}}(\mathrm{C}(\Lambda)^\circ)$ is a stable ∞-category with a t-structure determined by

$$(\mathsf{Sp}(\Lambda)_{\leq 0}, \mathsf{Sp}(\Lambda)_{\geq 0}).$$

Here we set

$$\mathsf{Sp}(\Lambda)_{\geq 0} := \mathsf{N}_{\mathrm{dg}}(\mathrm{C}(\Lambda))_{\geq 0} \cap \mathsf{D}_\infty(\Lambda), \quad \mathsf{Sp}(\Lambda)_{\leq 0} := \mathsf{N}_{\mathrm{dg}}(\mathrm{C}(\Lambda))_{\leq 0} \cap \mathsf{D}_\infty(\Lambda),$$

where $\mathsf{N}_{\mathrm{dg}}(\mathrm{C}(\Lambda))_{\geq n}$ (resp. $\mathsf{N}_{\mathrm{dg}}(\mathrm{C}(\Lambda))_{\leq n}$) denotes the full sub-$\infty$-category of $\mathsf{N}_{\mathrm{dg}}(\mathrm{C}(\Lambda))$ spanned by Λ-dg-modules M such that $H_k(M) \simeq 0$ for $k < n$ (resp. $k > n$).

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