Okounkov's conjecture on qMZV

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Abstract and contents

A quick review of

[Ok] A. Okounkov, "Hilbert schemes and multiple q-zeta values", 13pp., arXiv:1404.3873,

focusing on the section 2: $\mathrm{Hilb}^n(S)$ and qMZV.

qMZV appear in Nekrasov-like functions.

Contents

§1 $\mathrm{Hilb}^n(S)$ and qMZV [1/4]

S: a non-singular projective surface over $\mathbb{C}$
$S^{[n]} := \mathrm{Hilb}^n(S)$: the Hilbert scheme of n-points on S ($n \in \mathbb{Z}_{\geq 0}$)
: roughly, "moduli space of n-points on S" (explained in §2)
: non-singular projective of dimension $2n$

[Göttsche[1]]

$$\sum_{n=0}^{\infty} q^n e(S^{[n]}) = (q;q)_{\infty}^{-e(S)}$$

with $e(X) :=$ Euler number of X. □

For generalization, LHS $= \sum_{n=0}^{\infty} q^n \int_{S^{[n]}} c_{2n}(T_{S^{[n]}})$
with $T_{S^{[n]}}$ the tangent bundle and c_{2n} the top Chern class,
and $\int_{S^{[n]}}$ is the integration of cohomology.

[1] L. Göttsche, "The Betti numbers of the Hilbert scheme of points on a smooth projective surface", Math. Ann. **286** (1990), 193–207.

Göttsche formula: $\sum_{n=0}^{\infty} q^n \int_{S^{[n]}} c_{2n}(T_{S^{[n]}}) = (q;q)_\infty^{-e(S)}$.
For line bundles L, M on S and $f \in \mathbb{Q}[\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_2, \ldots]$,

$$\langle f \rangle_{L,M} := \sum_{n=0}^{\infty} q^n \int_{S^{[n]}} f(L^{[n]}) c_{2n}(T_{S^{[n]}}(M)) \in H^*(\mathrm{pt}; \mathbb{Q})[\![q]\!] = \mathbb{Q}[\![q]\!]$$

with $L^{[n]}$ and $T_{S^{[n]}}(M)$ certain vector bundles on $S^{[n]}$ (p.10, §2.4).

[Carlsson-Okounkov[2], Cor. 1], [Ok, §2.4]
$\langle 1 \rangle_M = \sum_{n=0}^{\infty} q^n \int_{S^{[n]}} c_{2n}(T_{S^{[n]}}(M))$ is equal to

$$\langle 1 \rangle_M = (q;q)_\infty^{-\delta}, \quad \delta := \int_S c_2(T_S \otimes M)$$

If $M = O_S$ (trivial bundle), then $T_S(M) = T_S$ and $\delta = e(S)$, recovering the Göttsche formula. □

[2] E. Carlsson, A. Okounkov, "Exts and vertex operators", Duke Math. J. **161** (2012), no. 9, 1797–1815.

§1 $\mathrm{Hilb}^n(S)$ and qMZV [3/4]

For $f \in \mathbb{Q}[\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_2, \dots]$ and line bundles L, M on S,
$\langle f \rangle_{L,M} := \sum_{n=0}^{\infty} q^n \int_{S^{[n]}} f(L^{[n]}) c_{2n}(T_{S^{[n]}}(M)) \in \mathbb{Q}[\![q]\!]$.

[Carlsson-Okounkov, Cor. 3], [Ok, §2.6]

$f = c_1 = \mathrm{ch}_1$: 1st Chern class, $L = O_S$,

$$\frac{\langle c_1 \rangle_{O_S,M}}{\langle 1 \rangle_M} = \frac{1}{2}(\widetilde{E}_q(2) - \widetilde{E}_q(3)) \cdot \int_S (c_1 c_2 - c_3)(T_S \oplus M)$$

with $\widetilde{E}_q(s) := \sum_{n>0} n^{s-1} \frac{q^n}{1-q^n}$. □

Relation to qMZV [Ok, §1]

$\zeta_q^O(s_1, \dots, s_k) := \sum_{n_1 > n_2 > \dots > n_k \geq 1} \prod_{i=1}^k (n_i)^{-s_i}$, $(n)^{-s} := \frac{p_s(q^n)}{(1-q^n)^s}$,
$p_s(t) := t^{s/2}$ (s: even), $t^{(s-1)/2}(t+1)$ (s: odd).
$q\mathrm{MZV} \subset \mathbb{Q}[\![q]\!]$: $\mathbb{Q}$-subalg. spanned by $\zeta_q^O(s_1, \dots, s_k)$ with $s_i \geq 2$.
$\widetilde{E}_q(2) = \zeta_q^O(2)$ and $\widetilde{E}_q(3) = \zeta_q^O(3)$ (but $\widetilde{E}_q(s) \neq \zeta_q^O(s)$ for $s \geq 4$) .

For $f \in \mathbb{Q}[\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_2, \ldots]$ and line bundles L, M on S,

$$\langle f \rangle_{L,M} := \sum_{n=0}^{\infty} q^n \int_{S^{[n]}} f(L^{[n]}) c_{2n}(T_{S^{[n]}}(M)) \in \mathbb{Q}[\![q]\!].$$

Normalized function

$$\langle f \rangle'_{L,M} := \langle f \rangle_{L,M} / \langle 1 \rangle_M.$$

Carlsson-Okounkov formula:

$$\langle c_1 \rangle'_{O_S,M} = \frac{1}{2}(\zeta_q^O(2) - \zeta_q^O(3)) \cdot \int_S (c_1 c_2 - c_3)(T_S \oplus M) \in q\mathrm{MZV}.$$

□

Conjecture 2 in [Ok]

For any $f \in \mathbb{Q}[\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_2, \ldots]$ and line bundles L, M on S,

$$\langle f \rangle'_{L,M} \overset{?}{\in} q\mathrm{MZV}.$$

□

Remark

If S quasi-projective with G-action, $\langle f \rangle'_{L,M} \in H^*_G(\mathrm{pt}; \mathbb{Q})_{\mathrm{loc}}[\![q]\!]$. □

§2.1 Hilbert scheme of points $\mathrm{Hilb}^n(X)$

[Grothendieck]

X: quasi-projective scheme over $\mathbb{C}$
$\exists\,\mathrm{Hilb}(X)$: scheme such that for all scheme T,

$$\mathrm{Hom}_{\mathrm{Sch}}(T, \mathrm{Hilb}(X)) = \mathcal{H}_X(T) := \left\{ Z \subset T \times X \;\middle|\; \begin{array}{l} \text{closed subscheme,} \\ \text{proper and flat } /T \end{array} \right\}.$$

It decomposes as $\mathrm{Hilb}(X) = \bigsqcup_{P \in \mathbb{Q}[\lambda]} \mathrm{Hilb}^P(X)$, and

$$\mathrm{Hom}_{\mathrm{Sch}}(T, \mathrm{Hilb}(X)) = \{ Z \in \mathcal{H}_X(T) \mid \Phi_{Z_t} = P \ \forall t \in T \}$$

with $\Phi_Y \in \mathbb{Q}[\lambda]$ the Hilbert polynomial. $\deg \Phi_Y = \dim Y$. □

In the case $P = n$ (constant, $n \in \mathbb{Z}_{\geq 0}$),

$$\mathrm{Hilb}^n(X) = \left\{ Z \subset X \mid \text{0-dim. subcsheme}, \dim_{\mathbb{C}} H^0(Z, O_Z) = n \right\}.$$

$\mathrm{Supp}(O_Z) = \{p_1, \ldots, p_l\}$, $p_i \in X$, $n = \sum_{i=1}^{l} \dim_{\mathbb{C}} O_{Z,p_i}$.

§2.2 Universal family and tangent space

$X^{[n]} := \mathrm{Hilb}^n(X)$. $\mathrm{Hom}_{\mathrm{Sch}}(T, X^{[n]}) = \mathcal{H}_X^n(T)$ for any scheme T.
$\mathcal{H}_X^n(T) := \{Z \subset T \times X \mid \text{closed sub., flat over } T, \Phi_{Z_t} = n\}$.

Universal family

It is a closed subscheme $\Sigma_n \subset X^{[n]} \times X$ given by

$$\mathrm{Hom}_{\mathrm{Sch}}(X^{[n]}, X^{[n]}) = \mathcal{H}_X^n(X^{[n]}), \quad \mathrm{id}_{X^{[n]}} \longmapsto \Sigma_n \subset X^{[n]} \times X$$

In fact, $\Sigma_n = \{(W, x) \in X^{[n]} \times X \mid x \in W\}$. □

Tangent space of $X^{[n]}$

For each $Z \in X^{[n]}$, the Zariski tangent space of $X^{[n]}$ at Z is

$$T_Z X^{[n]} = \mathrm{Hom}_{O_X}(I_Z, O_Z).$$

I_Z is the ideal sheaf. $0 \to I_Z \to O_X \to O_Z \to 0$ (exact). □

§2.3 Hilbert scheme of points on a surface

[Fogarty[3]]

For a non-singular (quasi-)projective surface S,
$S^{[n]}$ is non-singular (quasi)-projective. □

K-theory class of tangent space [Ok, §2.2]

Since $S^{[n]}$ is non-singular, the Zariski tangent space is equal to the fiber of the tangent bundle $T_{S^{[n]}}$ of the complex manifold $S^{[n]}$.
We have $T_{S^{[n]}}|_Z = T_Z S^{[n]} = \mathrm{Hom}_{O_S}(I_Z, O_Z)$.
Then, by [Ellingsrud-Göttsche-Lehn[4], Prop. 2.2],

$$T_{S^{[n]}}|_Z = \chi(O_S, O_S) - \chi(I_Z, I_Z)$$

in $K(\mathrm{pt})$, where $\chi(E, F) := \sum_i (-1)^i \mathrm{Ext}^i_{O_S}(E, F)$. □

[3] J. Fogarty, "Algebraic families on an algebraic surface", Am. J. Math., **10** (1968), 511-521.

[4] G. Ellingsrud, L. Göttsche, M. Lehn, "On the cobordism class of the Hilbert scheme of a surface", J. Algebraic Geom. **10** (2001), 81–100.

§2.4 Bundles in [Ok]

Recall the Nekrasov-like function

$$\langle f \rangle_{L,M} := \sum_{n=0}^{\infty} q^n \int_{S^{[n]}} f(L^{[n]}) c_{2n}(T_{S^{[n]}}(M)).$$

The bundle $L^{[n]}$ in [Ok]

L: a line bundle on S, $\Sigma_n \subset S^{[n]} \times S$: the universal family,
$p\colon \Sigma_n \to S^{[n]}$, $q\colon \Sigma_n \to S$: projections
$L^{[n]} := p_* q^* L$: vector bundle on $S^{[n]}$ of rank n.
The fiber at $Z \in S^{[n]}$ is $L^{[n]}\big|_Z = H^0(Z, O_Z \otimes_{O_X} L)$. □

The bundle $T_{S^{[n]}}(M)$ in [Ok]

M: a line bundle on S
We can twist the tangent bundle $T_{S^{[n]}}$ by M to get $T_{S^{[n]}}(M)$
such that $T_{S^{[n]}}(M)|_Z = \chi(O_S, M) - \chi(I_Z, I_Z \otimes M)$. □

§3.1 Carlsson-Okounkov vertex operator [1/3]

Consider the direct sum of the cohomology of $S^{[n]}$:

$$\mathcal{F} = \bigoplus_{n\geq 0} \mathcal{F}_n, \quad \mathcal{F}_n := H^*(S^{[n]}; \mathbb{Q}),$$

a linear superspace with $\mathcal{F}^{\mathrm{ev}} := \bigoplus_n H^{\mathrm{ev}}(S^{[n]}; \mathbb{Q})$, similar for $\mathcal{F}^{\mathrm{od}}$.

[Nakajima[5], Grojnowski[6]]

For each $n \in \mathbb{Z} \setminus \{0\}$ and $\gamma \in H^*(S; \mathbb{Q})$, there is $\alpha_n(\gamma) \in \mathrm{End}(\mathcal{F})$ such that

$$[\alpha_m(\gamma), \alpha_n(\gamma')] = (-1)^{m-1} m \delta_{m+n,0} \langle \gamma, \gamma' \rangle$$

with $\langle \gamma, \gamma' \rangle := \int_S \gamma \cup \gamma'$, and $\mathcal{F}$ is isomorphic to the Fock representation of this Heisenberg algebra. □

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[5] H. Nakajima, "Heisenberg algebra and Hilbert schemes of points on projective surfaces", Ann. Math. (2) **145** (1997), 379–388.

[6] I. Grojnowski, "Instantons and affine algebras I", Math. Res. Lett. **3** (1996), 275–291.

§3.1 Carlsson-Okounkov vertex operator [2/3]

[Carlsson-Okounkov[7]]

$\Sigma \subset S^{[n]} \times S$: the universal family, M: a line bundle, $k, l \in \mathbb{Z}_{\geq 0}$.
$\Sigma^{(i)} := p_{i3}^!(O_\Sigma) \in K(S^{[k]} \times S^{[l]} \times S)$ for $i = 1, 2$,
where $p_{13}\colon S^{[k]} \times S^{[l]} \times S \to S^{[k]} \times S$ is the proj., similar for p_{23},
and $p_{i3}^!$ is the K-theoretic pullback.

$$E_M := p_{12!}\big((\Sigma^{(1)\vee} + \Sigma^{(2)} - \Sigma^{(1)\vee} \cdot \Sigma^{(2)}) \cdot p_3^! M\big) \in K(S^{[k]} \times S^{[l]}),$$

where $p_!$ is the K-theoretic push-forward., $\vee$ is the K-theor. dual,
and $\cdot$ is the multiplication ($\otimes$ of bundles).
$\operatorname{rk} E_M = k + l$, and the fiber at $(Z_1, Z_2) \in S^{[k]} \times S^{[l]}$ is

$$E_M|_{(Z_1,Z_2)} = \chi(O_S, M) - \chi(I_{Z_1}, I_{Z_2} \otimes M).$$

So $E_M|_{\text{diag. of } S^{[k]} \times S^{[k]}} = T_{S^{[n]}}(M)$.

[7] E. Carlsson, A. Okounkov, "Exts and vertex operators", Duke J. M. (2012).

An operator series (quantum field) $W(M,z) \in (\mathrm{End}\,\mathcal{F})[\![z^{\pm 1}]\!]$:

$$\langle W(M,z)\xi,\eta\rangle = z^{l-k}\int_{S^{[k]}\times S^{[l]}} p_1^*(\xi)\cup p_2^*(\eta)\cup c_{k+l}(E_M)$$

for $\xi \in H^*(S^{[k]};\mathbb{Q})$ and $\eta \in H^*(S^{[l]};\mathbb{Q})$.

Theorem [Carlsson-Okounkov]

$W(M,z)$ has the following vertex operator expression:

$$\begin{aligned} W(M,z) = &\exp\Big(-\textstyle\sum_{n>0}\frac{1}{n}(-z)^n\alpha_{-n}(M)\Big)\\ &\cdot\exp\Big(-\textstyle\sum_{n>0}\frac{1}{n}z^{-n}\alpha_n(K_S-M)\Big), \end{aligned}$$

where $\alpha_n(M) := \alpha_n(c_1(M))$, $\alpha_n(K_S - M) := \alpha_n(c_1(K_S\otimes M^*))$ and K_S is the canonical bundle of S. □

Nekrasov-like function as a super-trace

$f \in \mathbb{Q}[\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_2, \dots]$, L, M: line bundles on S

$$\begin{aligned}\langle f\rangle_{L,M} &:= \sum_{n=0}^{\infty} q^n \int_{S^{[n]}} f(L^{[n]}) c_{2n}(T_{S^{[n]}}(M)) \\ &= \sum_{n=0}^{\infty} q^n \int_{S^{[n]}} f(L^{[n]}) c_{2n}(E_M|_{\text{diag. of } S^{[n]} \times S^{[n]}}) \\ &= \mathrm{str}_{\mathcal{F}}\big(q^{\mathrm{N}} \cdot f(L^{[\mathrm{N}]}) \cdot W(M,z)\big).\end{aligned}$$

$\mathrm{str}_{\mathcal{F}} := \mathrm{tr}_{\mathcal{F}^{\mathrm{ev}}} - \mathrm{tr}_{\mathcal{F}^{\mathrm{od}}}$, and $\mathrm{N}|_{\mathcal{F}_n} = n\,\mathrm{id}_{\mathcal{F}_n}$ is the number-of-points operator. □

§3.2 Computing Nekrasov-like function [2/4]

$\langle 1\rangle_M = \sum_{n=0}^{\infty} q^n \int_{S^{[n]}} c_{2n}(T_{S^{[n]}}(M))$ [CO, Cor.1], [Ok, §2.4]

Decompose $W(M,z) = \Gamma_-(z)\Gamma_+(z)$.

$$
\begin{aligned}
\langle 1\rangle_M &= \mathrm{str}\big(q^{\mathsf{N}}\Gamma_-(z)\Gamma_+(z)\big) && \\
&= \mathrm{str}\big(\Gamma_-(qz)q^{\mathsf{N}}\Gamma_+(z)\big) && \text{[commutation rel.]} \\
&= \mathrm{str}\big(q^{\mathsf{N}}\Gamma_+(z)\Gamma_-(qz)\big) && \text{[cycle prop. of str]} \\
&= (1-q)^{(M,K_S-M)}\,\mathrm{str}\big(q^{\mathsf{N}}\Gamma_-(qz)\Gamma_+(z)\big) && \text{[comm. rel. of VO]} \\
&= \cdots = (q;q)_\infty^{(M,K_S-M)}\,\mathrm{str}\big(q^{\mathsf{N}}\Gamma_+(z)\big) && \\
&= (q;q)_\infty^{(M,K_S-M)}\,\mathrm{str}\big(q^{\mathsf{N}}\big) && \text{[}\Gamma_+\text{ is triangular]} \\
&= (q;q)_\infty^{(M,K_S-M)}(q;q)_\infty^{-e(S)} && \text{[Göttsche formula]} \\
&= (q;q)_\infty^{-\int_S c_2(T_S\otimes M)}.
\end{aligned}
$$

□

Computing $\langle c_1 \rangle'_{O_S,M}$ [CO, Cor.3], [Ok, §2.6]

$$\langle c_1 \rangle'_{O_S,M} = \mathrm{str}\big(q^{\mathrm{N}} \cdot c_1(O_S^{[\mathrm{N}]}) \cdot W(M,z)\big).$$

Reduce to the case $S = \mathbb{C}^2 \curvearrowleft T = (\mathbb{C}^\times)^2$, $\mathcal{F} = \bigoplus_n H_T^*(S;\mathbb{Q})$.
t_1, t_2: T-equiv. weight of T_S, m: T-equiv. weight of O_S.

[Lehn[8]]

As an operator on the Fock space $\mathcal{F}$,

$$c_1(O_S^{[\mathrm{N}]}) = -\tfrac{1}{2}(t_1+t_2)\textstyle\sum_{k>0}(k-1)\alpha_{-k}\alpha_k + \tfrac{1}{2}t_1 t_2 \sum_{k,l>0} \alpha_{-k}\alpha_{-l}\alpha_{k+l} - \tfrac{1}{2}\sum_{k,l>0}\alpha_{-k-l}\alpha_k\alpha_l$$

with $\alpha_k := \alpha_k(1)$ [quantum Calogero-Sutherland operator]. □

Then one can compute

$$\langle c_1 \rangle'_{O_S,M} = \cdots = -\tfrac{1}{2}(t_1+t_2)\textstyle\sum_k k(k-1)q^k/(1-q^k) - \frac{1}{2t_1t_2}(t_1+t_2)(m+t_1+t_2)m \cdot \cdot\big[\sum_k q^k/(1-q)^k + \sum_{k,l} q^{k+l}/(1-q^k)(1-q^l)(1-q^{k+l})\big].$$

[8] M. Lehn, "Chern classes of tautological sheaves on Hilbert schemes of points on surfaces", Invent. Math. **136** (1999), 157–207.

§3.2 Computing Nekrasov-like function [4/4]

$\langle c_1 \rangle'_{O_S,M} = -A \sum_k - B(\sum_k + \sum_{k,l})$.
$A := \frac{1}{2}(t_1 + t_2)$, $B := \frac{1}{2t_1 t_2}(t_1 + t_2)(m + t_1 + t_2)m$.

Lemma

1st $\sum_k = \widetilde{E}_3(q) - \widetilde{E}_2(q)$, 2nd $\sum_k = q\frac{d}{dq}\widetilde{E}_1(q) - \widetilde{E}_2(q)$,
$\sum_{k,l} = \widetilde{E}_3(q) - q\frac{d}{dq}\widetilde{E}_1(q)$
with $\widetilde{E}_s(q) := \sum_{n>0} n^{s-1}q^n/(1-q^n)$. □

Hence $\langle c_1 \rangle'_{O_S,M} = (A+B)(\widetilde{E}_2(q) - \widetilde{E}_3(q))$, but
$A + B = (t_1 + t_2)(t_1 + m)(t_2 + m)/2t_1 t_2 = \frac{1}{2}\int_S (c_1 c_2 - c_3)(T_S \oplus M)$
in $H^*_T(S = \mathbb{C}^2; \mathbb{Q})$. Thus

$$\langle c_1 \rangle'_{O_S,M} = \frac{1}{2}(\widetilde{E}_2(q) - \widetilde{E}_3(q)) \cdot \int_S (c_1 c_2 - c_3)(T_S \oplus M).$$

□