0. Introduction

[FP] Fulton, Pandharipande, "Notes on stable maps and quantum cohomology",
Proc. Symp. Pure Math. 62, AMS, 1997; arXiv:alg-geom/9608011

Contents

§1 – §6: Definition, construction and properties of the moduli space of stable maps

§7 – §10: Quantum cohomology

I will explain

I. Stable maps and the moduli space $\bar{M}_{g,n}(X,\beta)$
II. Outline of construction → [R2]
III. Theorems of [FP] on $\bar{M}_{0,n}(X,\beta)$ (especiially for $X = G/P$) → [R4], [R5]
IV. Basics of quantum cohomology → [R1], [R3]

I.1. Moduli functor of n points in $\mathbb{P}^1$

Schemes are over $\mathbb{C}$.

Consider the classification of $\{(p_1, \ldots, p_n) \mid p_i \in \mathbb{P}^1, p_i \neq p_j \ \forall i \neq j\}$ under the equivalance $(p_1, \ldots, p_n) \sim (q_1, \ldots, q_n) \ \Leftrightarrow \exists\, \Phi \in \mathrm{Aut}(\mathbb{P}^1) \text{ s.t. } \Phi(p_i) = q_i \ \forall i\ \}$.
The corresponding moduli space would be the "classifying space" of the equiv. classes.

[Grothendieck]

One should consider not only the objects but the families of such objects over any base.

A family of n points on $\mathbb{P}^1$ over a base scheme B is $(\pi, \sigma_1, \ldots, \sigma_n)$ with

- A (flat proper) morphism $\pi: X \to B$ of schemes such that $\pi^{-1}(b) \in \mathbb{P}^1 \ \ \forall b \in B$
- n disjoint sections $\sigma_i: B \to X$

Two families $(\pi: X \to B, \sigma_i)$ and $(\pi': X' \to B, \sigma_i{}')$ are equivalent if $\exists\, \Phi: X \to X'$ isom. s.t. ...

The moduli functor $\mathcal{M}_{0,n}$: $Schemes \to Sets^{op}$ assigns to a scheme B the set of classes and to any morphism $f: B' \to B$ the map induced by the publlback of families.

Ex. For $\mathrm{pt} = \mathrm{Spec}(\mathbb{C})$, $\mathcal{M}_{0,n}(\mathrm{pt}) = \{$equivalence classes $[p_1, \ldots, p_n]$ in the beginning$\}$.

I.2. Moduli space of n points in ℙ^1

A fine moduli $M_{0,n}$ is a scheme representing $\mathcal{M}_{0,\mathrm{n}}$, i.e., ∃ fun. isom. $\mathcal{M}_{o,n} \to \mathrm{Hom}(-, M_{0,n})$.

For $n = 3$, $\mathcal{M}_{0,3}(\mathrm{pt}) = \mathrm{pt} = \{(0,1,\infty)\}$ by $\mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}_2(\mathbb{C})$.

Given any family $(\pi: X \to B, \sigma_1, \sigma_2, \sigma_3)$, consider the morphism $T: X \to B \times \mathbb{P}^1$ defined by

$\mathrm{T(x)} \coloneqq \left(\pi(\mathrm{x}), \mathrm{CrossRatio}_{\pi(\mathrm{x})}\left(\sigma_1(\pi(\mathrm{x})), \sigma_2(\pi(\mathrm{x})), \sigma_3(\pi(\mathrm{x})), \mathrm{x}\right)\right)$, $\mathrm{CR}(\mathrm{p}_1, \dots, \mathrm{p}_4) \coloneqq \mathrm{p}_{14}\mathrm{p}_{23}/\mathrm{p}_{12}\mathrm{p}_{34}$

It yields an isomorphism of families $(\pi, \sigma_1, \dots)$ to $(\pi_1: B \times \mathbb{P}^1 \to B, 0,1,\infty)$.

Thus, we have the fine moduli $\mathrm{M}_{0,3} = \mathrm{pt}$.

For $\mathrm{n} = 4$, we have the fine moduli $\mathrm{M}_{0,4} = \mathbb{P}^1 - \{0,1,\infty\}$, since any $(\mathrm{p}_1, \dots, \mathrm{p}_4)$ is equivalent to $(0,1,\infty, \mathrm{CR}(\mathrm{p}_1, \dots, \mathrm{p}_4))$.

For $\mathrm{n} \geq 4$, we have the fine moduli $\mathrm{M}_{\mathrm{o,n}} = \left(\mathrm{M}_{0,4}\right)^{\mathrm{n}-3} - (\text{all diagonals})$

[since any $(\mathrm{p}_1, \dots, \mathrm{p}_4)$ is equivalent to $(0,1,\infty, \mathrm{CR}(\mathrm{p}_1, \mathrm{p}_2, \mathrm{p}_3, \mathrm{p}_4), \dots, \mathrm{CR}(\mathrm{p}_1, \mathrm{p}_2, \mathrm{p}_3, \mathrm{p}_\mathrm{n}))$.]

For $\mathrm{n} \leq 2$, $\mathcal{M}_{0,\mathrm{n}}(\mathrm{pt})$ consists of only one point, but there is some B with non-equiv. families.

There is no fine moduli.

I.3. Moduli functor and fine moduli space

Generally, a moduli functor means a functor $\mathcal{F}: Schemes \to Sets^{op}$.
Similarly, a fine moduli of $\mathcal{F}$ is defined.

For a fine moduli space M of $\mathcal{F}$ and functor isom $\varphi: \mathcal{F} \to \mathrm{Hom}(-, M)$,
the universal family is defined to be $\varphi^{-1}(id_M) \in \mathcal{F}(M)$,
which can be expressed as a family $\pi: U \to M$ of objects over the base M.
It has the property that $\forall m \in M = \mathcal{F}(\mathrm{pt})$ corresponding to the equivalence classes of objects $[X_m]$, we have $[\pi^{-1}(m)] = [X_m]$.

For the fine moduli $\mathrm{M}_{0,3} = \mathrm{pt}$, the universal family is $\mathrm{U}_{0,3} = (\pi: \mathbb{P}^1 \to \mathrm{pt}, 0,1,\infty)$.

For the fine moduli $\mathrm{M}_{0,4} = \mathbb{P}^1 - \{0,1,\infty\}$, the universal family is
$\mathrm{U}_{0,4} = \left(\pi_1: \mathrm{M}_{0,4} \times \mathbb{P}^1 \to \mathrm{M}_{0,4}, 0,1,\infty,\delta\right)$ with δ the diagonal section $\delta(\mathrm{p}) = \mathrm{p}$.

I.4. Rational stable curves

Remark. Grassmann functor $\mathcal{GR}: Schemes \to Sets^{op}$,

$\mathcal{GR}(S) = \{\text{equivalence classes } [\mathcal{F}, q] \text{ of quotients } q: \mathcal{O}_S^{\oplus n} \to \mathcal{F} \text{ with } \mathcal{F} \text{ locally free of rank } k\}$

has a fine moduli, nothing but the (quotient) Grassmannian $\mathrm{Gr}^{\mathrm{quot}}(\mathbb{C}^{\mathrm{n}}, \mathrm{k})$.

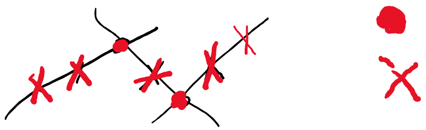
The universal family $\pi: \mathcal{Q} \to \mathrm{Gr}^{\mathrm{quot}}(\mathbb{C}^n, k)$ is the tautological quotient bundle $\mathcal{Q}$.

[Special case of Quot scheme]

$\mathrm{M}_{0,\mathrm{n}}$ is open, and one wants to compactify it.

It would be nice if the universal family is also lifted to the compactification. A solution is:

An n-pointed rational stable curve $(C, p_1, \ldots, p_n)$ consists of a connected curve C with at worst nodal singularity and n-tuple of distinct points in $C - \mathrm{Sing}(C)$ such that for each irreducible component $E \subset C$, there are at least 3 special points on E.



: singular points
: marked points
special points

Nodal singularity: locally $(xy = 0)$ in $\mathbb{C}^2 = \mathrm{Spec}\ \mathbb{C}[x, y]$.

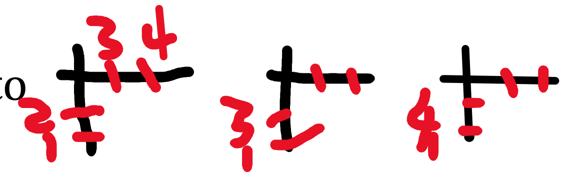
The condition means $\mathrm{Aut}(C, \{p_i\})$ is a finite group (no non-trivial infinitesimal aut.).

I.5. Moduli of pointed curves and KDM moduli

Thm. [Knudsen]

There exists an irreducible smooth projective fine moduli space $\bar{M}_{0,\mathrm{n}}$ for n-pointed rational stable curves which contains $M_{0,n}$ as open dense sub.

The boundary $\partial \mathrm{M}_{0,\mathrm{n}} \coloneqq \bar{M}_{0,\mathrm{n}} - M_{0,n}$ consisting of nodal stable curves is NCD.

Ex. $\bar{M}_{0,\mathrm{n}} = \mathbb{P}^1$, $\partial \mathrm{M}_{0,4} = \{0,1,\infty\}$, corresponds to 

$\partial \mathrm{M}_{0,5}$ consists of 10 codim 1 strata and 15 codim 2 strata

Higher genus case [Deligne-Mumford, Knudsen]

An n-pointed stable curve $(C, p_1, \ldots, p_n)$ of genus g is a pair of a connected curve C with at worst nodal sing. and n-tuple of distinct points in $C - \mathrm{Sing}(C)$ such that for $E \subset C$,

- If $E \cong \mathbb{P}^1$, then E must contain at least 3 special points.
- If $\mathrm{g}(E) = 1$, then E must contain at least 1 special point.

These conditions are equivalent to $\mathrm{Aut}(C, \{p_i\})$ is a finite group.

In this case, we have the moduli space $\bar{M}_{g,n}$ as DM stack (orbifold) of $\dim = 3g - 3 + \mathrm{n}$.

II.1. Stable maps

An n-pointed genus g quasi-stable curve $(C, p_1, \dots, p_n) = (C, \{p_i\})$ is a projective connected reduced curve C of arithmetic **genus** g with at most nodal singularities, equipped with n-distinct non-singular marked points $p_1, \dots, p_n$.

X: scheme (of finite type over $\mathbb{C}$)

A map from a quasi-stable curve $(C, \{p_i\})$ to X is a morphism $f: C \to X$ of schemes, denoted as $(C, \{p_i\}, f)$.

Such a map is stable if for each irreducible component E of C,

- If $E \cong \mathbb{P}^1$ and $f(E)$ is a point, then E must contain at least 3 special points.
- If $\mathrm{g}(E) = 1$ and $f(E)$ is a point, then E must contain at least 1 special point.

Equivalent to $\mathbf{Aut}(\mathrm{C}, \{\mathrm{p_i}\}, \mathrm{f})$ is a finite group (no non-trivial infinitesimal aut.)

II.2. Kontsevich moduli

X: projective variety

The Chow (co)homology in even deg: $\mathrm{A_d}(X) \coloneqq \mathrm{H_{2d}}(X,\mathbb{Z})$, $\mathrm{A^d}(X) \coloneqq \mathrm{H^{2d}}(X,\mathbb{Z})$.

For $\beta \in \mathrm{A_1}(X) = \mathrm{H_2}(X,\mathbb{Z})$, the moduli functor $\mathcal{M}: Sch \to Sets^{op}$,

$\mathcal{M}(S) \coloneqq \{\text{isom clasess of stable maps } (C \to S, \{p_i\}, f: C \to X) \text{ over } S \textit{ with } f_*[C] = \beta\}$.

FP's Thm. 2, proved in §4, §5:

The functor $\mathcal{M}$ has a coarse moduli space $\bar{M} = \bar{M}_{\mathrm{g,n}}(X,\beta)$ as a projective scheme.

It is a compactification of the moduli $M_{\mathrm{g,n}}(\mathrm{X},\beta)$ of nonsingular domains C.

For $\mathcal{F}: Sch \to Sets^{\mathrm{op}}$, the coarse moduli M is a scheme with fun. morph. $\varphi: \mathcal{F} \to \mathrm{Hom}(-,M)$

such that $\varphi(\mathrm{Spec}\,\mathbb{C})$ is a bijection and

$$\begin{array}{ccc} \mathcal{M} & \longrightarrow & \mathrm{Hom}(-,M) \\ & {\scriptstyle\forall\psi}\searrow \quad \circlearrowleft & \downarrow {\scriptstyle\exists!\,\alpha} \\ & & \mathrm{Hom}(-,\forall \mathrm{N}) \end{array}$$

Kontsevich [1995, PM129] sketched the construction as a Deligne-Mumford stack.

III. Construction of moduli

Remark 1. Trivial case $\beta = 0$.

$\bar{M}_{g,n}(X, 0) = \bar{M}_{g,n} \times X$, $\bar{M}_{g,n}$: Deligne-Mumford moduli (orbifold in general).

Remark 2. $\bar{M}_{0,n}(\mathbb{P}^n, 1) = G(\mathbb{P}^1, \mathbb{P}^r) = \mathrm{Gr}(2, r+1)$

Outline of the construction

§3 A rigidification of $\bar{M}_{g,n}(\mathbb{P}^r, d)$: $d \in \mathrm{A}_1(\mathbb{P}^\mathrm{r}) = \mathrm{H}_2(\mathbb{P}^\mathrm{r}, \mathbb{Z}) = \mathbb{Z}H \cong \mathbb{Z}$

for a basis $\bar{t} \subset V^* := H^0(\mathbb{P}^r, \mathcal{O}(1))$,

construct a coarse moduli space $\bar{M}_{g,n}(\mathbb{P}^r, d, \bar{t})$ of $\bar{t}$-rigid stable maps.

$\bar{M}_{0,n}(\mathbb{P}^r, d, \bar{t})$ is a nonsingular algebraic variety.

§4 The construction of $\bar{M}_{g,n}(\mathbb{P}^r, d)$: gluing $\{\bar{M}_{g,n}(\mathbb{P}^r, d, \bar{t}) \mid \bar{t} \subset V^*: \text{basis}\}$

§5 The construction of $\bar{M}_{g,n}(X, \beta)$:

for projective variety X, $\iota: \mathrm{X} \hookrightarrow \mathbb{P}^\mathrm{r}$, $\iota*(\beta) = dH$,

construct $\bar{M}_{g,n}(X, \beta, \bar{t}) \subset \bar{M}_{g,n}(\mathbb{P}^r, d, \bar{t})$ and glue them.

IV.1. Genus 0 Moduli spaces

A nonsingular projective variety is convex if $\forall f: \mathbb{P}^1 \to X,\ f^*(T_X)$ is globally generated.

A homogeneous space $X = G/P$ is an example. ($\mathrm{T_X}$ globally generated,

FP's Thm. 2, proved in §3 - §5:

X: nonsingular projective convex variety, $\beta \in A_1(X)$, $\bar{M} := \bar{M}_{0,n}(X, \beta)$.

(1) $\bar{M}$ is a normal projective variety of pure dimension $\dim X - 3 + \int_\beta c_1(T_X) + n$.

(2) $\bar{M}$ is locally a quotient of a nonsingular variety by a finite group.

(3) $\bar{M}^* \subset \bar{M}$: the open locus of stable maps with no non-trivial automorphism.

$\bar{M}^*$ is a nonsingular fine moduli space of the corresponding functor $\mathcal{M}^*$.

FP's Thm. 3, proved in §6: used in [R4], [R5]

X: as in Thm. 2, $\beta \in A_1(X)$, $\bar{M} := \bar{M}_{0,n}(X, \beta)$.

$\bar{M} \supset \partial\bar{M} := \{f: C \to X \mid C \text{ is reducible}\}$ is a divisor with normal crossings.

IV.2. Dimension of moduli space

The argument of dimension in [§ 5.2, FM] seems to be inaccurate.
The following is based on § 7.1.4 of
Cox, Katz, "Mirror Symmetry and Algebraic Geometry", MSM 68, AMS, 1998.

For $\bar{M} = \bar{M}_{g,n}(X,\beta)$, and its point $m \in \bar{M}$ corresponding to $(C, \{p_i\}, f: C \to X)$,

$$\mathrm{exp.dim_m}\, \bar{M} \coloneqq \mathrm{dim(1st\ order\ deformations\ of}\ m) - \mathrm{dim(obstructions\ for}\ m)$$

$$\text{(1st order deformations)} = \mathrm{Ext}^1_C(f^*\Omega^1_X \to \Omega^1_C(\textstyle\sum_{i=1}^n p_i), O_C), \quad \text{(obstructions)} = \mathrm{Ext}^2_C(\dots, O_C),$$

$$0 \to \mathrm{Ext}^0_C(\dots, O_C) \to \mathrm{Ext}^0_C(\Omega^1_C(\dots), O_C) \to \mathrm{Ext}^0_C(f^*\Omega^1_X, O_C) \to \mathrm{Ext}^1_C(\dots, O_C) \to \cdots$$

3rd arrow is injective by the stability of f, so $\mathrm{Ext}^0_C(\dots, O_C) = 0$.

$$\therefore\ \mathrm{exp.dim} = -\chi(\dots, O_C) = -\chi(\Omega^1_C(\dots), O_C) + \chi(f^*\Omega^1_X, O_C) = \mathrm{ext}^1 - \mathrm{ext}^0 + \chi(f^*\Theta_X)$$

(at the last equality, used $\mathrm{Ext}^i_C(\Omega^1_C(\dots), O_C) \cong H^i(f^*\Theta_X)$)

HRR gives $\chi(f^*\Theta_X) = -\int_\beta \omega_X + (1-g)\dim X$.

$$-\int_\beta \omega_X := \omega_X(\beta) \quad \text{for} \quad \beta \in A_1(X) = H_2(X,\mathbb{Z}), \quad \omega_x = -c_1(T_X) \in H^2(X,\mathbb{Z}).$$

$$\mathrm{ext}^1 - \mathrm{ext}^0 = \big(\text{1st order deform. of } (C, \{p_i\})\big) - (\text{infinitesimal aut.}) = \dim \bar{M}_{g,n} = 3g - 3 + n.$$

Therefore $\mathrm{exp.dim_m}\, \bar{M} = (1-g)(\dim X - 3) - \int_\beta \omega_X + n$.

V.1 Gromov-Witten invariant

X: nonsingular projective convex variety (e.g. homogeneous variety G/P)

$\beta \in A_1(X)$, $\bar{M} := \bar{M}_{0,n}(X,\beta) \supset M := M_{0,n}(X,\beta)$,

Evaluation maps: $\mathrm{ev}_1, \dots, \mathrm{ev}_n : \bar{M} \to X$, $(C, \{p_i\}, f : C \to X) \mapsto f(p_i)$.

$\gamma_1, \dots, \gamma_n \in A^*(X)$ gives rise to cohomology class $\mathrm{ev}_1^*(\gamma_1) \cup \cdots \cup \mathrm{ev}_n^*(\gamma_n) \in A^*(\bar{M})$.

Gromov-Witten invariant: $I_\beta(\gamma_1, \dots, \gamma_n) := \int_{[\bar{M}]} \mathrm{ev}_1^*(\gamma_1) \cup \cdots \cup \mathrm{ev}_n^*(\gamma_n)$.

It is 0 unless $\sum \mathrm{codim}\, \gamma_i = \dim \bar{M} = \dim X - 3 + \int_\beta c_1(T_X) + n$ (for homog. γ_i).

It is symmetric for $\gamma_1, \dots, \gamma_n$.

FP's Lem.14 in §7: used in [R3]

$X = G/P$, $g_1, \dots, g_n \in G$: general elements, $\Omega_1, \dots, \Omega_n \subset X$: pure dimensional subvar.

The scheme theoretic intersection $Z := \mathrm{ev}_1^{-1}([g_1\Omega_1]) \cap \cdots \cap \mathrm{ev}_n^{-1}([g_n\Omega_n])$

is a finite number of reduced points supported in M, and $I_\beta([\Omega_1], \dots, [\Omega_n]) = \# Z$.

V.2 Quantum cohomology

$X = G/P$, $\beta \in A_1(X)$, $\bar{M} \coloneqq \bar{M}_{0,n}(X,\beta)$ as before.

$T_0 \coloneqq 1 \in A^0(X)$, $T_1, \ldots, T_p$: basis of $A^1(X)$, $T_{p+1}, \ldots, T_m$: the other basis of $A^*(X)$.

The classical cup product is given by GW invariants at $\beta = 0$:

$\bar{M}_{0,n}(X, \beta = 0) = \bar{M}_{0,n} \times X.$

$\Delta \subset X \times X$: the diagonal, $[\Delta] = \sum g^{ef} T_e \otimes T_f \in A^*(X \times X) = A^*(X) \otimes A^*(X)$.

$T_i \cup T_j = \pi_{2*}\left([\Delta] \cup \pi_1^*(T_i \cup T_j)\right) = \sum(\int_X T_i \cup T_j \cup T_e)\, g^{ef} T_f = \sum I_0(T_i T_j T_e) g^{ef} T_f.$

Quantum cup product:

$$T_i * T_j \coloneqq \sum_{ef} \Phi_{ije} g^{ef} T_f, \quad \Phi_{ijk} \coloneqq \sum_{n\geq 0} \sum_\beta \frac{1}{n!} I_\beta(\gamma^n T_i T_j T_k), \quad \gamma \coloneqq \sum_{i=0}^m y_i T_i.$$

FP's Thm. 4 in §7 (big quantum cohomology ring):

The operation $*$ makes $A^*(X) \otimes \mathbb{Q}[[y]]$ into a comm. unital assoc. $\mathbb{Q}[[y]]$-algebra.

V.3 Small quantum cohomology ring

$X = G/P$, $\beta \in A_1(X)$, $\bar{M} \coloneqq \bar{M}_{0,n}(X,\beta)$ as before.

$T_0 \coloneqq 1 \in A^0(X)$, $T_1, \dots, T_p$: basis of $A^1(X)$.

Small quantum product:

$$\bar{\Phi}_{ijk} \coloneqq \Phi_{ijk}(y_0, \dots, y_p, 0, \dots, 0) = \int_X T_i \cup T_j \cup T_k + \bar{\Gamma}_{ijk},$$

$$\bar{\Gamma}_{ijk} \coloneqq \sum_{n \geq 0} \frac{1}{n!} \sum_{\beta \neq 0} I_\beta\left(\bar{\gamma}^n T_i T_j T_k\right) \in A^*(X) \otimes \mathbb{Z}[q_1, \dots, q_p], \qquad q_i \coloneqq e^{y_i}.$$

$$T_i * T_j \coloneqq \textstyle\sum_{ef} \bar{\Phi}_{ije} g^{ef} T_f = T_i \cup T_j + \sum_{ef} \bar{\Gamma}_{ije} g^{ef} T_f .$$

We have a unital commutative associative $\mathbb{Z}[q]$-algebra [R1]

$\mathrm{QH}_s^*(\mathrm{X}) \coloneqq (A^*(X) \otimes \mathbb{Z}[q], *)$: the small quantum cohomology ring of X.