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§4 Affine root systems

type GL $x = (x_1, \dots, x_n)$, $q, t \in \mathbb{C}$,

$P_\lambda(x) = P_\lambda(x; q, t) \in \mathbb{C}[x]$ $\quad \ell(\lambda) \le n$

(∗) $D_x^1 P_\lambda(x) = P_\lambda(x) \varepsilon_\lambda$ $\qquad \varepsilon_\lambda = e_1(q^\lambda t^\delta) \in \mathbb{C}$

$$D_x^1 := \sum_{i=1}^n \prod_{j \ne i} \frac{1 - t x_j / x_i}{1 - x_j / x_i} T_{q, x_i}$$

$\lambda = \phi$: $P_\lambda(x) = 1$

$$(*) \iff \sum_{i=1}^n \prod_{j \ne i} \frac{1 - t x_j / x_i}{1 - x_j / x_i} = e_1(t^\delta) = t^{n-1} + \dots + 1$$

$$\iff \sum_{w \in \mathfrak{S}_n} \prod_{1 \le i < j \le n} \frac{1 - t x_{w(j)} / x_{w(i)}}{1 - x_{w(j)} / x_{w(i)}} = \mathrm{cst}.$$

(indep. of x)

[Macdonald. 1972. Math. Ann.]

R : irreducible finite root system ← [Bourbaki]

$\Pi \subset R$: positive roots

$W_{\mathrm{fin}} = W(R)$: Weyl group.

$$\sum_{w \in W_{\mathrm{fin}}} \prod_{\alpha \in \Pi} \frac{1 - t_\alpha e^{-w\alpha}}{1 - e^{-w\alpha}} = \sum_{w \in W_{\mathrm{fin}}} \prod_{\alpha \in \Pi \cap w^{-1}(-\Pi)} t_\alpha$$

Hecke param. ⟶ $t : \Pi \to \mathbb{C}$, $\alpha \mapsto t_\alpha$, s.t. $\alpha \in W\beta \Rightarrow t_\alpha = t_\beta$

(∗) is recovered by

$R = A_{n-1} = \{ \alpha_{i,j} := \varepsilon_i - \varepsilon_j \mid 1 \le i \ne j \le n \} \subset \bigoplus_{i=1}^n \mathbb{C}\varepsilon_i$

$\Pi = \{ \alpha_{ij} \mid i < j \}$, $\quad x_i = e^{\varepsilon_i}$, $\quad t_\alpha = t \ (\forall \alpha \in \Pi)$

⇝ D_x's and P_λ's for root systems ...

Another motivation :

type GL $q = 0$: $P_\lambda(x; 0, t) =$ Hall-Littlewood polynom.

$=$ zonal sph. func. of (GL_n, K_λ) over $\mathbb{Q}_p$

$(t = 1/p)$

$\exists$ 3 formulations of root systems for Macdonald polynomials

① admissible pair (R, S) of finite root systems
[Macdonald 1987 preprint: arXiv:0011046]

② pair (S, S') of affine root systems
[Macdonald 2003 book] "Affine root systems & orthogonal polynomials"

③ initial data $(R, \Delta, \circ, \Lambda, \Lambda^\vee)$ ($\circ = u$ (untwist) or t (twist))
[Haiman 2006 PICM] [Stokman, in vol. 5 of Askey-Bateman project. arXiv:1111.6112]

② $V = \mathbb{R}^n$: linear space w/ std. inner product $\langle \cdot, \cdot \rangle$, $|v| := \langle v, v \rangle^{\frac{1}{2}}$

$E = \mathbb{R}^n$: Euclidean space with metric $d(x, y)$

: affine space $/\mathbb{R}$ (faithful & transitive action by add. gp. V)

$x, y \in E$, $v \in V \Rightarrow x + v \in E$, $x - y \in V$, $d(x,y) = |x - y|$

$F := \{ f : E \to \mathbb{R} \mid$ affine-linear maps between affine spaces $\}$

$= \{$ " $\mid$ $\exists Df : V \to \mathbb{R}$ linear func. $f(x+v) = f(x) + (Df)(v)$. $\forall x \in E$, $\forall v \in V \}$

: $\mathbb{R}$-lin. sp., $D : F \to V^*$ linear, onto,

$F = V^* \oplus \ker D = V^* \oplus \mathbb{R}c$ $\quad c(v) = 1 \ \forall v \in V$

- $\langle \cdot, \cdot \rangle$ on F : $\langle f, g \rangle := \langle Df, Dg \rangle$
- $v \in V \setminus \{0\}$ $\quad v^\vee := 2v / |v|^2 \in V$

$f \in F \setminus \mathbb{R}c$ $\quad f^\vee := 2f / |f|^2 \in F$

$s_f : E \to E$, isometry

$s_f(x) := x - f^\vee(x) \cdot Df$ $\quad (x \in E)$

$H_f := f^{-1}(0) \subset E$
codim $= 1$

<u>Dfn</u> affine root sys. on E [Macdonald. 1972. Inv.]
: subset $S \subset F$ s.t.

(AR1) $\langle S \rangle_{\mathbb{R}\text{-lin.}} = F$, $S \cap \mathbb{R}c = \phi$
(AR2) $\forall a, b \in S$ $s_a(b) \in S$
(AR3) 〃 $\langle a^\vee, b \rangle \in \mathbb{Z}$
(AR4) $W_S := \langle s_a \mid a \in S \rangle_{\text{grp}} \subset \mathrm{Isom}(E)$
$W_S \curvearrowright E$ proper ($K, K' \subset E$ cpt. $\#\{w \in W_S \mid wK \cap K' \neq \phi\} < \infty$) □

(AR3) implies: $a, \lambda a \in S$ $(\lambda \in \mathbb{R}) \Rightarrow \lambda \in \{\pm\frac{1}{2}, \pm 1, \pm 2\}$ $(\ast)$
$\rightsquigarrow$ <u>Dfn.</u> S is reduced if $\lambda = \pm 1$ $\forall a \in S$ □

<u>Dfn.</u> S is irreducible if $\nexists S_1, S_2 \subset S, \neq \phi$, s.t. $\langle a_1, a_2 \rangle = 0$ $\forall a_i \in S_i$ □

S: irreducible affine root sys
$\{H_a \mid a \in S\}$ is locally finite
alcove := conn. comp. of $E \setminus \bigcup_{a \in S} H_a$: open rectilinear n-simplex
$W_S \curvearrowright \{\text{alcoves}\}$,
C: alcove $B(C) := \left\{ a \in S \;\middle|\; \begin{array}{l} \text{indivisible } [\lambda = \pm 1 \text{ in } (\ast)] \\ H_a \text{ is a wall of } C \\ \forall x \in C \;\; a(x) > 0 \end{array} \right\}$

$\Rightarrow \# B(C) = n+1$, $B(C)$ is a linear basis of F

<u>Dfn.</u> $B \subset S$ is a basis of S if $\exists$ chamber $B = B(C)$ □

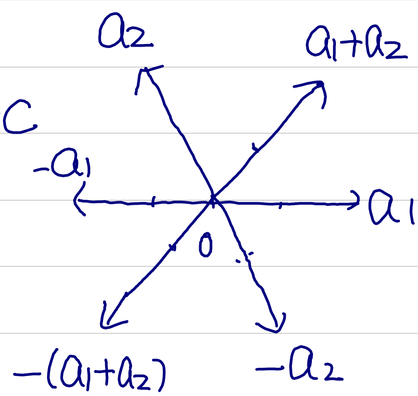
<u>E.g.</u> $S = S(A_n)$: $E = \{{}^t(x_1, \ldots x_{n+1}) = \sum_{i=1}^{n+1} x_i e_i \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} x_i = 0\}$.
$S := \{\pm(e_i - e_j) + rc \mid 1 \le i < j \le n+1, r \in \mathbb{Z}\}$
$B = \{a_0 := -\varepsilon_1 + \varepsilon_{n+1} + c, \; a_i := e_i - e_{i+1} \; (1 \le i \le n)\}$

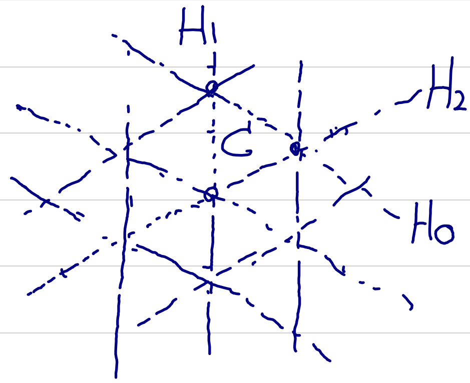
$(n=2)$

$a_0 = -e_1 + e_3 + c$

$a_1 = e_1 - e_2$

$a_2 = e_2 - e_3$





$a_1, a_2 \in H_0$

Claim. R: irreducible finite root sys. [type $A_n - G_2$, BC_n] ← non-reduced

V: $\mathbb{R}$-lin. sp., spanned by R, w/ inner prd. $\langle \cdot, \cdot \rangle$

$\Rightarrow$ $\forall \alpha \in R$, $\forall r \in \mathbb{Z}$ $\quad a_{\alpha,r} : V \to \mathbb{R}$, $a_{\alpha,r}(x) := \langle \alpha, x \rangle + r$

is aff. linear, and

$S(R) := \{ \underset{\alpha + rc}{\underset{\parallel}{a_{\alpha,r}}} \mid \alpha \in R, r \in \mathbb{Z} \}$ is an irred. aff. root sys. $\square$

Dynkin diagram of S

Choose $B = \{ a_i \mid i = 0, \dots, n \}$: basis of S

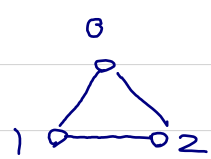
vertex set $I := \{0, \dots, n\}$

$n_{ij} := \langle a_i^\vee, a_j \rangle \quad (i, j \in I)$

$d_{ij} := \max(|n_{ij}|, |n_{ji}|)$

$\#\{\text{edges between } i \text{ and } j\} = d_{ij}$

$\underset{i}{\circ} \to \underset{j}{\circ}$ if $|a_i| > |a_j|$



$S(A_2)$

classification of irreducible root sys. (see [Macdonald 2003, §1.3])

Dfn. & Claim $S^\vee := \{a^\vee \in F \mid a \in S\}$: aff. root sys. □

reduced $\Rightarrow$ $S = S(R)$ or $S(R)^\vee$, R : irred fin. root sys. of type X ($A_n - G_2, BC_n$)

called type	X		$X^\vee$
	$\boxed{A_{n\geq 1}}$	$B_{n\geq 3}$	$B_n^\vee$
	$\boxed{D_{n\geq 4}}$	$C_{n\geq 2}$	$C_n^\vee$
	$\boxed{E_{6-8}}$	F_4	$F_4^\vee$
	$\boxed{BC_{n\geq 1}}$	G_2	$G_2^\vee$

$\because S(R)^\vee \cong S(R)$

non-reduced $\Rightarrow$ $S = S_1 \cup S_2$, $W_{S_1} \cong W_{S_2}$

$S_1 := \{a \in S \mid \tfrac{1}{2}a \notin S\}$] reduced irred.

$S_2 := \{a \in S \mid 2a \notin S\}$] aff. root systems.

called type (X, Y) X: type of S_1, Y: type of S_2

$(X, Y) = (C_n^\vee, C_n)$

$(C_2, C_2^\vee)$, $(B_n, B_n^\vee)$ $n \geq 3$

$(C_n^\vee, BC_n)$ $n \geq 1$

(BC_n, C_n) $n \geq 1$

subsys. of $(C_n^\vee, C_n)$

of W_S-orbits (= # of Hecke parameters)

#	reduced	non-red
1	A, D, E,	
2	$B, B^\vee, F, F^\vee, G, G^\vee, BC_1$	
3	$C, C^\vee, BC_{n\geq 2}$	others
4		$(C_1^\vee, C_1)$ Askey-Wilson
5		$(C_n^\vee, C_n)_{n\geq 2}$ Koornwinder