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§3. Macdonald polynomials of type A

$q, t \in \mathbb{C}$, $|q|<1$, $x=(x_1,\dots,x_n)$

Macdonald q-difference operators (of type A)

$r=1,\dots,n$ $\quad D_x^{(r)}(q,t) := \sum_{\substack{I\subset\{1,\dots,n\}\\ |I|=r}} \prod_{\substack{i\in I\\ j\notin I}} \frac{1-tx_i/x_j}{1-x_i/x_j} \prod_{i\in I} T_{q,x_i}$

<u>Fact.</u> [Macdonald. 1987]

(1) $[D_x^{(r)}, D_x^{(s)}]=0$

(2) $D_x^{(r)} \curvearrowright \mathbb{C}[x^{\pm}]^{\mathfrak{S}_n}$ $\qquad \{\lambda=(\lambda_1,\dots,\lambda_n)\in\mathbb{Z}^n \mid \lambda_1\geq\cdots\geq\lambda_n\}$

(3) $\exists!$ basis $\{P_\lambda(x;q,t) \mid \lambda\in P_+\}$ of $\mathbb{C}[x^{\pm}]^{\mathfrak{S}_n}$

s.t. for $\lambda\in P_+\cap\mathbb{N}^n$ (partitions of length $\leq n$)

$$\begin{cases} P_\lambda \in m_\lambda + \sum_{\mu<\lambda}\mathbb{C}\cdot m_\mu \\ D_x^{(r)}P_\lambda(x) = P_\lambda(x)\,\varepsilon_\lambda^{(r)} \quad \exists\, \varepsilon_\lambda^{(r)}\in\mathbb{C} \end{cases}$$

← 文献：野海，東北大集中講義．(1997)

§4. Macdonald operators and extended affine Hecke alg.

$\mathfrak{S}_n = W_{fin} = \langle s_1, \ldots, s_{n-1} \mid \ \rangle$

$\subset W_{aff} = \langle s_0, \ldots, s_{n-1} \mid \ \rangle$

$\subset W = \langle \omega, s_0, \ldots, s_{n-1} \rangle \qquad \omega s_i = s_{i-1}\omega \quad (i \in \mathbb{Z}/n\mathbb{Z})$

$W_{aff} \cong Q \rtimes W_{fin}. \qquad Q := \sum_{i=1}^{n-1} \mathbb{Z}\alpha_i \subset V = \bigoplus_{i=1}^{n} \mathbb{R}\varepsilon_i$

: root lattice of type A $\qquad \alpha_i = \varepsilon_i - \varepsilon_{i+1}$

$W_{fin} \curvearrowright Q, V. \qquad \sigma.\varepsilon_i = \varepsilon_{\sigma(i)}$

$\mathfrak{S}_n \ni \sigma$

$W \cong P \rtimes W_{fin}, \qquad P := \sum_{i=1}^{n} \mathbb{Z}\varepsilon_i$: weight lattice $\quad P \supset Q$

Dfn. (Lusztig operator) $\quad i = 0, \ldots, n-1 \qquad \mathbb{C}(x_1, \ldots, x_n) = \mathbb{C}(x)$

$$T_i := t^{-1/2} \cdot \frac{1 - t x^{\alpha_i}}{1 - x^{\alpha_i}} s_i + \frac{t^{1/2} - t^{-1/2}}{1 - x^{\alpha_i}} \curvearrowright \mathbb{C}(x)$$

- $x^{\alpha_i} := \begin{cases} x^{\varepsilon_i - \varepsilon_{i+1}} = x_i / x_{i+1} & (i = 1, \ldots, n-1) \\ x^{\delta + \varepsilon_n - \varepsilon_1} = q x_n / x_1 & (i = 0) \end{cases}$
- $s_i(x_j) = x_{s_i(j)} \quad (i \geq 1), \quad s_0 = (1, n)\, T_{q,x_1} \cdot T_{q,x_n}^{-1}$ □

Fact. [Lusztig 1989] $\quad \widetilde{\omega} := s_{n-1} \cdots s_1 T_{q,x_1}$

$\mathrm{End}(\mathbb{C}(x)) \supset \langle T_0, \ldots, T_{n-1}, \widetilde{\omega} \rangle_{alg.} \cong H(W)$: extended aff. Hecke alg. (of type A)

$H(W)$: generated by $T_0, \ldots, T_{n-1}, \widetilde{\omega}^{\pm 1}$

fund. rel. $\begin{cases} (T_i - t^{1/2})(T_i + t^{-1/2}) = 0 \\ T_i T_j = T_j T_i \\ T_i T_j T_i = T_j T_i T_j \\ \omega T_i = T_{i-1} \omega \end{cases}$ □

Thm. $H(W) \ni Y_i \quad (i = 1, \ldots, n) \quad Y_1 := T_1 T_2 \cdots T_{n-1} \widetilde{\omega}, \ Y_2 := T_2 \cdots T_{n-1} \widetilde{\omega} T_1^{-1}, \ldots,$

$Y_n := \widetilde{\omega} T_1^{-1} \cdots T_n^{-1}$

$\Rightarrow Z(H(W)) = \mathbb{C}[Y_1, \ldots, Y_n],$

$D_x^{(r)}(q,t) = \sum_{1 \leq i_1 < \cdots < i_r \leq n} Y_{i_1} \cdots Y_{i_r}$ on $\mathbb{C}(x)^{\mathfrak{S}_n}$ □

$T = c s + d \qquad s : x \mapsto x^{-1} \qquad c, d \in \mathrm{Fun}(x)$

$(T-\alpha)(T-\beta) = 0 \qquad \alpha, \beta$: scalar

$\Rightarrow (T-\alpha).1 = 0$ [or $(T-\beta).1 = 0$]

$\Rightarrow \quad c + d = \alpha, \quad T - \alpha = c(s-1), \quad T - \beta = c(s-1) + (\alpha - \beta)$

$\Rightarrow \quad 0 = c(s-1)(cs + \alpha - \beta - c) = c(s(c) - (\alpha - \beta - c))(1-s)$

$\Rightarrow \quad c + s(c) = \alpha - \beta, \qquad d = \alpha - c = s(c) + \beta$

Conversely, if $c(x)$ satisfies $c(x) + c(x^{-1}) = \alpha - \beta$ $(*)$
then $T := c(x) \cdot s + d(x)$, $d(x) := c(x^{-1}) + \beta$
satisfies $(T-\alpha)(T-\beta) = 0$

E.g. $c(x) := (\alpha + \beta x)/(1-x)$, $\alpha = t^{1/2}$, $\beta = -t^{-1/2}$
(c-function of p-adic spherical functions)