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§2 Ruijsenaars operators

$x=(x_1,\dots,x_n)\in\mathbb{C}^n$, $f(x)=f(x_1,\dots,x_n)$ shift param

$T_{q,x_i}f(x):=f(x_1,\dots,qx_i,\dots,x_n)$ $q\in\mathbb{C},\ |q|<1$

$\theta(z;p):=(z;p)_\infty(p/z;p)_\infty$ $p\in\mathbb{C},\ |p|<1$ ell. param

$(z;p)_\infty:=\prod_{i=1}^{\infty}(1-p^iz)$

$t\in\mathbb{C}$ Hecke param.

$r=1,\dots,n$ $D_x^{(r)}(q,t,p):=\sum_{\substack{I\subset\{1,\dots,n\}\\|I|=r}}\prod_{\substack{j\in I\\ i\notin I}}\frac{\theta(tx_i/x_j;p)}{\theta(x_i/x_j;p)}\prod_{i\in I}T_{q,x_i}$

Thm. [Ruijsenaars 1987]

$$[D_x^{(r)},D_x^{(s)}]=0\quad \forall r,s=1,\dots,n$$

□

§3. Macdonald polynomials of type A

$q, t \in \mathbb{C}$, $|q|<1$,

$x=(x_1,\dots,x_n)$ $\qquad P_+ := \{\lambda=(\lambda_1,\dots,\lambda_n)\in\mathbb{Z}^n \mid \lambda_1\geqq\cdots\geqq\lambda_n\}$

T_{q,x_i} : <u>q-shift operator</u> $(T_{q,x_i}f)(x) := f(x_1,\dots,qx_i,\dots,x_n)$

<u>Macdonald q-difference operators</u> (of type A_{n-1})

$r=1,\dots,n$ $\qquad D_x^{(r)}(q,t) := \sum_{\substack{I\subset\{1,\dots,n\}\\ |I|=r}} \prod_{\substack{i\in I\\ j\notin I}} \frac{1-tx_i/x_j}{1-x_i/x_j} \prod_{i\in I} T_{q,x_i}$

$= \lim_{p\to 0} D_x^{(r)}(q,t,p)$

$r=1$: $\quad D_x^{(1)}(q,t) = \sum_{i=1}^{n} \left(\prod_{j\neq i} \frac{1-tx_i/x_j}{1-x_i/x_j}\right) T_{q,x_i}$

$r=2$: $\quad D_x^{(2)}(q,t) = \sum_{1\leq i_1<i_2\leq n} \prod_{j\neq i_1,i_2} \frac{1-tx_{i_1}/x_j}{1-x_{i_1}/x_j}\frac{1-tx_{i_2}/x_j}{1-x_{i_2}/x_j} T_{q,x_{i_1}}T_{q,x_{i_2}}$

<u>Fact</u>. [Macdonald. 1987]

(1) $[D_x^{(r)}, D_x^{(s)}]=0$ $\qquad$ (mutually commutative)

(2) $D_x^{(r)} \curvearrowright \mathbb{C}[x^{\pm1}]^{\mathfrak{S}_n}$ $\qquad$ (preserving sym. Laurent poly.)

(3) $\exists!$ basis $\{P_\lambda(x;q,t) \mid \lambda\in P_+\}$ of $\mathbb{C}[x^{\pm1}]^{\mathfrak{S}_n}$

s.t. for $\lambda\in P_+\cap\mathbb{N}^n=\{\text{partitions}\}$,

$$\begin{cases} P_\lambda \in m_\lambda + \sum_{\mu<\lambda}\mathbb{C}\cdot m_\mu \\ D_x^{(r)} P_\lambda(x) = P_\lambda(x)\,\varepsilon_\lambda^{(r)} \quad \exists\, \varepsilon_\lambda^{(r)}\in\mathbb{C} \end{cases}$$

$P_\lambda(x;q,t)$: <u>Macdonald polynomials of type A_{n-1}</u>

or <u>" symmetric polynomials</u> $\square$

<u>Rmk</u>. $\varepsilon_\lambda^{(r)} = e_r(q^\lambda t^\rho) = e_r(q^{\lambda_1}t^{n-1}, q^{\lambda_2}t^{n-2},\dots,q^{\lambda_n})$ $\square$

$\uparrow$ elementary sym. polynom. of n-variables.

Fact. [Macdonald] Another characterization: $\begin{cases} \text{triangular.} & P_\lambda \in m_\lambda + \sum_{\mu < \lambda} \mathbb{C} m_\mu \\ \text{orthogonal}: & (P_\lambda, P_\mu)_{q,t} \propto \delta_{\lambda,\mu} \end{cases}$

$$(f, g)_{q,t} := \frac{1}{n!} \int_T f(x) g(x) W_{q,t}(x)\, dx$$

$$W_{q,t}(x) := \prod_{1 \le i \ne j \le n} \frac{(x_i/x_j ; q)_\infty}{(t x_i/x_j ; q)_\infty} \qquad \square$$

$$(x ; q)_\infty := (1-x)(1-qx)(1-q^2 x) \cdots$$

Rmk. $W_{q,t=q}(x) = \prod_{i \ne j} (1 - x_i/x_j) = W_1(x)$

$$\lim_{q \to 1} W_{q,t=q^\beta}(x) = \prod_{i \ne j} (1 - x_i/x_j)^\beta = W_\beta(x) \qquad \square$$

Cor. $P_\lambda(x ; q, t = q) = S_\lambda(x)$

$$\lim_{q \to 1} P_\lambda(x ; q, t = q^\beta) = J_\lambda(x) \qquad \square$$

E.g. $P_{(1^k)} = m_{(1^k)}$

$$P_{(2)} = m_{(2)} + \frac{(1+q)(1-t)}{1-qt} m_{(1^2)},$$

$$P_{(3)} = m_{(3)} + \frac{(1-q^3)(1-t)}{(1-q)(1-q^2 t)} m_{(2,1)} + \frac{(1-q^2)(1-q^3)(1-t)^2}{(1-q)^2(1-qt)(1-q^2 t)} m_{(1^3)}$$

$$P_{(2,1)} = m_{(2,1)} + (1-t)(2+q+t+2qt)/(1-qt^2) \cdot m_{(1^3)} \qquad \square$$