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Macdonald polynomials

§1 Schur and Jack polynomials

$x = (x_1, \dots, x_n)$: commuting variables

$s_\lambda(x) = s_\lambda(x_1, \dots, x_n)$: Schur polynom. of $\lambda \in P_+$ ← dominant weights of A_n

$P_+ := \{\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n \mid \lambda_1 \geq \dots \geq \lambda_n\}$

$$s_\lambda(x) = \frac{\det(x_i^{\lambda_j + n - j})}{\det(x_i^{n-j})_{i,j=1}^n} = \frac{\begin{vmatrix} x_1^{\lambda_1+n-1} & x_1^{\lambda_2+n-2} & \cdots & x_1^{\lambda_n} \\ x_2^{\lambda_1+n-1} & x_2^{\lambda_2+n-2} & \cdots & x_2^{\lambda_n} \\ \vdots & & & \vdots \end{vmatrix}}{\begin{vmatrix} x_1^{n-1} & x_1^{n-2} & \cdots & x_1^0 \\ x_2^{n-1} & x_2^{n-2} & & x_2^0 \\ \vdots & & & \end{vmatrix}}$$

$\prod_{i<j}(x_i - x_j)$

$\cap$

$\mathbb{Z}[x^{\pm 1}]^{\mathfrak{S}_n} = \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]^{\mathfrak{S}_n}$

sym. Laurent poly.

- For $\lambda \in P_+ \cap \mathbb{N}^n = \{$partitions of length $\leq n\}$,

  $s_\lambda(x) \in \mathbb{Z}[x]^{\mathfrak{S}_n} = \mathbb{Z}[e_1, \dots, e_n]$: sym. polynom.

  $e_m = \sum_{1 \leq i_1 < \dots < i_m \leq n} x_{i_1} \cdots x_{i_m} = s_{(1^m)}$

  $h_m = \sum_{1 \leq i_1 \leq \dots \leq i_m \leq n}$ 〃 $= s_{(m)}$

- $P_+ \ni \mu = \lambda - (\ell^n)$, $\exists \lambda \in P_+ \cap \mathbb{N}^n$, $\ell \in \mathbb{N}$

  $s_\mu = (x_1 \cdots x_n)^{-\ell} s_\lambda = e_n^{-\ell} s_\lambda$

  $\in \mathbb{Z}[e_1, \dots, e_n, e_n^{-1}] = \mathbb{Z}[x^{\pm 1}]^{\mathfrak{S}_n}$

- $\{s_\lambda(x) \mid \lambda \in P_+\}$ is a basis of $\mathbb{Z}[x^{\pm 1}]^{\mathfrak{S}_n}$

- triangular expansion of s_λ for $\lambda \in P_+ \cap \mathbb{N}$

  $s_\lambda = m_\lambda + \sum_{\mu < \lambda} c_{\lambda\mu} m_\mu$, $\exists c_{\lambda\mu} \in \mathbb{Z}$

  $m_\mu := \sum_{\nu \in \mathfrak{S}_n . \mu} x^\nu = \sum_\nu x_1^{\nu_1} \cdots x_n^{\nu_n}$

  $n=3$. $m_{(2,1,0)} = x_1^2 x_2^1 x_3^0 + x_1^2 x_2^0 x_3^1 + x^{(120)} + x^{(102)} + x^{(021)} + x^{(012)}$

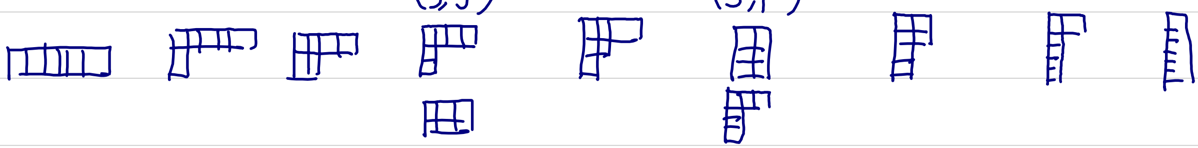
  $\{m_\mu \mid \mu \in P_+ \cap \mathbb{N}^n\}$: basis of the free $\mathbb{Z}$-mod. $\mathbb{Z}[x]^{\mathfrak{S}_n}$

$\mu \leqq \lambda \;:\Leftrightarrow\; |\mu|_n = |\lambda|_n$ & $|\mu|_k \leqq |\lambda|_k \quad \forall k=1,\dots,n-1$

dominance ordering $\qquad |\mu|_k := \mu_1 + \cdots + \mu_k$

$(2)>(1^2)$, $(3)>(2,1)>(1^3)$, $(4)>(3,1)>(2^2)>(2,1^2)>(1^4)$, $(5)>(4,1)>(3,2)>(3,1^2)>(2^2,1)>(2,1^3)>(1^5)$

$$(6)>(5,1)>(4,2) \begin{smallmatrix} >(4,1,1)> \\ >(3,3)> \end{smallmatrix} (3,2,1) \begin{smallmatrix} >(2^3)> \\ >(3,1^3)> \end{smallmatrix} (2^2,1^2)>(2,1^4)>(1^6)$$



total order for $|\lambda| \leqq 5$, partial order for $|\lambda| \geqq 6$

$h_2 \quad p_2 = x_1^2+x_2^2 \quad e_2 = x_1x_2$

$S_{(1)} = m_{(1)}$, $S_{(2)} = m_{(2)} + m_{(1^2)}$, $S_{(1^2)} = m_{(1^2)} = e_2$

$S_{(3)} = m_{(3)} + m_{(2,1)} + m_{(1^3)}$, $S_{(2,1)} = m_{(2,1)} + 2m_{(1^3)}$, $S_{(1^3)} = m_{(1^3)}$

- orthogonality $\quad \lambda,\mu \in P_+ \quad (S_\lambda, S_\mu)_1 = \delta_{\lambda,\mu}$

$$(f,g)_1 := \frac{1}{n!}\int_T f(x)\, g(x)\, w_1(x)\, dx$$

$$T = \{(x_1,\dots,x_n) \in \mathbb{C}^n \mid |x_i| = 1\} \quad dx = \prod_{i=1}^n \frac{dx_i}{2\pi\sqrt{-1}\,x_i}$$

$$w_1(x) := \prod_{i\neq j}(1 - x_i/x_j)$$

- For $\lambda \in P_+ \cap \mathbb{N}$, S_λ is the unique element of $\mathbb{Z}[x]^{S_n}$ s.t. $\begin{cases} S_\lambda \in m_\lambda + \sum_{\mu<\lambda} \mathbb{Z} m_\mu \\ (S_\lambda, S_\mu)_1 \propto \delta_{\lambda,\mu} \end{cases}$

$P_\lambda(x;\beta) \in \mathbb{Q}(\beta)[x]^{\mathfrak{S}_n}$: Jack polynom. of $\lambda \in \mathcal{P}_+$ with parameter β

- $P_\lambda(x;\beta=1) = s_\lambda(x)$: β-deformation of Schur
- uniquely characterized by
  $$\begin{cases} P_\lambda(\beta) \in m_\lambda + \sum_{\mu<\lambda} \mathbb{Q}(\beta) m_\mu \\ (P_\lambda, P_\mu)_\beta \propto \delta_{\lambda,\mu} \quad \text{with} \end{cases}$$
  $$(f,g)_\beta := \frac{1}{n!} \int_T \overline{f(x)}\, g(x)\, w_\beta(x)\, dx$$
  $$w_\beta(x) := \prod_{i\neq j} (1 - x_i/x_j)^\beta$$
- $P_{(1)} = m_{(1)}$, $P_{(2)} = m_{(2)} + \frac{2\beta}{1+\beta} m_{(1^2)}$, $P_{(1^2)} = m_{(1^2)}$
  $P_{(3)} = m_{(3)} + \frac{3\beta}{2+\beta} m_{(2,1)} + \frac{6\beta^2}{(1+\beta)(2+\beta)} m_{(1^3)}$
  $P_{(2,1)} = m_{(2,1)} + \frac{6\beta}{1+2\beta} m_{(1^3)}$, $P_{(1^3)} = m_{(1^3)}$

- another characterization
  $$\begin{cases} P_\lambda(\beta) \in m_\lambda + \sum_{\mu<\lambda} \mathbb{Q}(\beta)\, m_\mu \\ H_{CS} \cdot P_\lambda(\beta) \in \mathbb{Q}(\beta) \cdot P_\lambda(\beta) \quad (\text{eigen-func. of } H_{CS}) \end{cases}$$
  $$H_{CS} := \sum_{i=1}^n \vartheta_i^2 + \beta \sum_{1\le i<j\le n} \frac{x_i + x_j}{x_i - x_j} (\vartheta_i - \vartheta_j)$$
  Calogero–Sutherland Hamiltonian $\quad \vartheta_i = x_i \frac{\partial}{\partial x_i}$
  eigenvalue $\epsilon_\lambda(\beta) = \sum_{i=1}^n (\lambda_i^2 + \beta(n+1-2i)\lambda_i)$
- H_{CS} belongs to ∞-family of commuting operators $O_1, \underset{\overset{\|}{H_{CS}}}{O_2}, O_3, \ldots$
  $O_k := \sum_{i=1}^n D_i(\beta)^k \qquad [O_k, O_\ell] = 0$ on $\mathbb{Q}(\beta)[x]^{\mathfrak{S}_n}$
  $D_i(\beta) := \vartheta_i + \beta \sum_{j\neq i} \frac{x_i}{x_j - x_i}(1 - s_{ij})$ Dunkl operator
- Jack polynomials are joint eigenfunctions of O_k's
  $$O_k \cdot P_\lambda(x) = P_\lambda(x)\, \epsilon_\lambda^{(k)}$$
  $$\epsilon_\lambda^{(2)} = \epsilon_\lambda(\beta)$$